FERMATEAN INTUITIONISTIC FUZZY n-NORMED LINEAR SPACES: CONVERGENCE, COMPLETENESS, AND FIXED POINT APPLICATIONS

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Abstract. This paper introduces and studies the concept of Fermatean intuitionistic fuzzy n-normed linear spaces. By combining the geometric structure of n-normed linear spaces with the cubic admissibility condition of Fermatean fuzzy sets, we define a new framework in which the membership and non-membership functions satisfy $\mu_F^3+\nu_F^3 \leq 1$. Several fundamental properties of the proposed space are investigated. We introduce the Fermatean hesitation degree, open balls, convergence, Cauchy sequences, and completeness in this setting. Equivalent characterizations of convergence and Cauchy sequences are established in terms of the limiting behavior of the Fermatean membership and non-membership functions. We also prove that the limit of a convergent sequence is unique and that every convergent sequence is Cauchy. Furthermore, Fermatean intuitionistic fuzzy contraction mappings are introduced by using cubic-control functions, and a Banach-type fixed point theorem is proved in complete Fermatean intuitionistic fuzzy n-normed linear spaces. Finally, an application is provided to guarantee the existence and uniqueness of solutions of a linear operator equation. Several examples are included to illustrate the validity and applicability of the proposed concepts.

1. Introduction

The study of uncertainty has become one of the central themes in modern mathematics, applied analysis, decision theory, optimization, control theory, and information sciences. Classical set theory describes membership in a crisp way: an element either belongs to a set or it does not. However, many real-world situations involve incomplete, vague, or imprecise information. To overcome this limitation, Zadeh introduced the theory of fuzzy sets, where each element is assigned a degree of membership in the interval $[0, 1]$ [1]. This idea opened a new direction in mathematical modelling, especially for systems in which uncertainty cannot be represented adequately by classical binary logic. The importance of fuzzy sets was further emphasized by several foundational works on fuzzy decision-making, fuzzy topological structures, and many-valued uncertainty models [2, 3, 4, 5].

A significant development of fuzzy set theory was introduced by Atanassov through the concept of intuitionistic fuzzy sets [6]. In this framework, each element is described not only by a membership degree but also by a non-membership degree. Thus, an intuitionistic fuzzy set provides a richer structure than an ordinary fuzzy set because it also incorporates the hesitation degree. Further refinements and extensions of intuitionistic fuzzy sets were studied in [7, 8, 9, 10]. Related models, such as vague sets, interval-valued intuitionistic fuzzy sets, and distance-based approaches to intuitionistic fuzzy sets, were also developed in order to

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handle different types of uncertainty [11, 12, 13, 14]. These works show that membership and non-membership functions provide a powerful way to represent incomplete information in mathematical structures.

The relationship between fuzzy set theory and functional analysis has also been widely investigated. Fuzzy metric spaces, fuzzy normed linear spaces, intuitionistic fuzzy metric spaces, and intuitionistic fuzzy normed spaces have been introduced and studied from several perspectives [25, 26, 27, 28, 29]. In these spaces, convergence, Cauchy sequences, completeness, continuity, boundedness, and fixed point properties are described through fuzzy membership and non-membership functions rather than through ordinary distances alone. The study of bounded linear operators and fixed point results in fuzzy normed spaces has also produced many useful results [31, 32, 33, 34, 35]. Furthermore, statistical convergence and related sequence concepts have been extended to fuzzy and intuitionistic fuzzy normed structures [36, 37].

On the other hand, the theory of n-normed linear spaces provides a natural generalization of normed and 2-normed spaces. The idea of a 2-normed space was introduced by Gähler, and later developments led to the study of n-normed spaces and related geometric structures [38, 39, 40]. In an n-normed linear space, the size of a vector is not measured independently but through a function depending on n vectors. This allows one to describe geometric quantities related to linear dependence, volume, and determinant-type structures. The development of fuzzy n-normed linear spaces and intuitionistic fuzzy n-normed linear spaces was carried out in [41, 42, 43]. These works introduced convergence, Cauchy sequences, completeness, and other basic analytical concepts in the setting of fuzzy and intuitionistic fuzzy n-normed spaces. Further contributions to statistical convergence and calculus-type results in intuitionistic fuzzy n-normed spaces can be found in [44, 45, 46].

Recent research has considerably broadened the study of convergence, summability, and structural properties in neutrosophic spaces. Deferred statistical summability in neutrosophic n-normed linear spaces was developed in [50], while completeness and Cartesian-product constructions in neutrosophic rectangular n-normed spaces were investigated in [51]. Further advances involving rough and deferred $\mathcal{I}$-statistical convergence of difference sequences were presented in [49, 54]. A broader treatment of recent developments in neutrosophic analysis, together with related topics in algebra, graph theory, and applications, is provided in [52, 53]. These contributions demonstrate the growing importance of ideal-based, deferred, and rough convergence methods in the analysis of sequences under uncertainty.

In recent years, orthopair fuzzy models have attracted considerable attention. Yager introduced Pythagorean fuzzy sets, which replace the intuitionistic condition

$$\mu + \nu \leq 1$$

by the weaker quadratic condition

$$\mu^2 + \nu^2 \leq 1.$$

This modification allows a wider range of admissible membership and non-membership values [17]. Yager later introduced generalized orthopair fuzzy sets, which further extend the admissible region by using a parameterized power condition [18]. Related developments in hesitant fuzzy sets and aggregation methods also contributed to the generalization of uncertainty modelling [19, 15, 16].

A more recent and important extension is the theory of Fermatean fuzzy sets introduced by Senapati and Yager [21]. In a Fermatean fuzzy set, the membership and non-membership degrees satisfy the cubic condition

$$\mu^3 + \nu^3 \leq 1.$$

This condition is weaker than the intuitionistic fuzzy condition $\mu + \nu \leq 1$ and also weaker than the Pythagorean fuzzy condition $\mu^2 + \nu^2 \leq 1$ in the sense that it allows a larger admissible region for membership and non-membership values. Because of this flexibility, Fermatean fuzzy sets have been applied to decision-making, aggregation operators, medical decision problems,

and uncertainty modelling [20, 22, 23, 24]. These studies show that Fermatean fuzzy models provide an effective framework for representing complex uncertain information.

To address these issues, this paper introduces the concept of a Fermatean intuitionistic fuzzy n-normed linear space, abbreviated as FIF-n-NLS. The proposed structure combines the geometric nature of n-normed linear spaces with the cubic admissibility condition of Fermatean fuzzy sets. More precisely, for each

$$\widetilde{V} = (v_1, v_2, \ldots, v_n) \in X^n$$

and $\lambda > 0$, the membership and non-membership functions

$$\mu_F, \nu_F : X^n \times (0, \infty) \to [0, 1]$$

are required to satisfy

$$\mu_F^3(\widetilde{V}, \lambda) + \nu_F^3(\widetilde{V}, \lambda) \leq 1.$$

This condition allows the proposed space to contain intuitionistic fuzzy n-normed structures as special or compatible cases, while also permitting a broader range of uncertainty.

The main purpose of this paper is to establish the foundational theory of FIF-n-NLS. We first recall preliminary notions such as continuous t-norms, continuous t-conorms, intuitionistic fuzzy normed linear spaces, and intuitionistic fuzzy n-normed linear spaces. We then introduce the formal definition of a Fermatean intuitionistic fuzzy n-normed linear space and define the associated Fermatean hesitation degree. Several examples are provided to show that the proposed definition is nonempty and mathematically consistent.

After defining the space, we study open balls, convergence, Cauchy sequences, and completeness. The convergence of a sequence $\langle x_k \rangle$ to a point $x \in X$ is characterized by the conditions

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) \to 1$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) \to 0$$

for every $\lambda > 0$ and every $z_1, z_2, \ldots, z_{n-1} \in X$. Similarly, a Cauchy sequence is characterized by the limiting behavior of

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda)$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda)$$

as $k, p \to \infty$. We prove that limits are unique, that every convergent sequence is Cauchy, and that completeness can be formulated naturally in this Fermatean intuitionistic fuzzy n-normed setting.

The final part of this paper is devoted to contraction mappings and fixed point theory. Fixed point theorems are important tools in nonlinear analysis and have been widely studied in metric, normed, fuzzy metric, and fuzzy normed spaces [48, 30, 32, 35, 47]. In the present setting, a contraction mapping must be described in terms of the Fermatean membership and non-membership functions. We introduce Fermatean intuitionistic fuzzy cubic-control functions and define a suitable class of Fermatean intuitionistic fuzzy contraction mappings. A Banach-type fixed point theorem is then established in complete FIF-n-NLS. Finally, an application is given to prove the existence and uniqueness of a solution of a linear operator equation of the form

$$x = c + \alpha x, \qquad |\alpha| < 1.$$

This application illustrates that the abstract theory developed in the paper can be used to solve concrete operator equations under Fermatean intuitionistic fuzzy n-normed assumptions.

2. Preliminary Concepts

In this section, we recall some basic notions that will be used throughout the paper. These preliminary concepts provide the foundation for constructing Fermatean intuitionistic fuzzy n-normed linear spaces. We begin with continuous t-norms and continuous t-conorms, which are used to combine membership and non-membership values in fuzzy and intuitionistic fuzzy structures. These operations play an important role in defining triangle-type conditions in fuzzy normed and intuitionistic fuzzy normed spaces.

We also review fuzzy sets, intuitionistic fuzzy sets, and Fermatean fuzzy sets. In a fuzzy set, each element is described only by a membership degree, whereas in an intuitionistic fuzzy set, each element is described by both a membership degree and a non-membership degree satisfying

$$\mu + \nu \leq 1.$$

In contrast, a Fermatean fuzzy set uses the cubic admissibility condition

$$\mu^3 + \nu^3 \leq 1.$$

This condition gives more flexibility in assigning membership and non-membership degrees.

Furthermore, we recall the definition of an n-normed linear space. In such a space, the norm depends on n vectors and is closely related to linear dependence and volume-type measurements. The usual determinant norm on $\mathbb{R}^n$ provides a standard example of an n-norm. We also recall ordinary convergence, Cauchy sequences, and completeness in n-normed linear spaces, since these ideas will later be extended to the Fermatean intuitionistic fuzzy n-normed setting.

Finally, we include the basic definition of a fixed point, which will be used in the development of contraction mappings and the fixed point theorem in complete Fermatean intuitionistic fuzzy n-normed linear spaces.

Definition 2.1. *A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if, for all $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0,1]$, the following conditions are satisfied:*

(1) $\varepsilon_1 * 1 = \varepsilon_1$;
(2) $\varepsilon_1 * \varepsilon_2 = \varepsilon_2 * \varepsilon_1$;
(3) $(\varepsilon_1 * \varepsilon_2) * \varepsilon_3 = \varepsilon_1 * (\varepsilon_2 * \varepsilon_3)$;
(4) *if* $\varepsilon_1 \leq \varepsilon_3$ *and* $\varepsilon_2 \leq \varepsilon_4$, *then*

$$\varepsilon_1 * \varepsilon_2 \leq \varepsilon_3 * \varepsilon_4;$$

(5) $*$ *is continuous on* $[0,1] \times [0,1]$.

Example 2.1. *Some standard examples of continuous t-norms are:*

(1) *Product t-norm:*

$$\varepsilon_1 * \varepsilon_2 = \varepsilon_1 \varepsilon_2.$$

(2) *Minimum t-norm:*

$$\varepsilon_1 * \varepsilon_2 = \min\{\varepsilon_1, \varepsilon_2\}.$$

(3) *Lukasiewicz t-norm:*

$$\varepsilon_1 * \varepsilon_2 = \max\{\varepsilon_1 + \varepsilon_2 - 1, 0\}.$$

Definition 2.2. *A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-conorm if, for all $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [0,1]$, the following conditions are satisfied:*

(1) $\varepsilon_1 \diamond 0 = \varepsilon_1$;
(2) $\varepsilon_1 \diamond \varepsilon_2 = \varepsilon_2 \diamond \varepsilon_1$;
(3) $(\varepsilon_1 \diamond \varepsilon_2) \diamond \varepsilon_3 = \varepsilon_1 \diamond (\varepsilon_2 \diamond \varepsilon_3)$;
(4) *if* $\varepsilon_1 \leq \varepsilon_3$ *and* $\varepsilon_2 \leq \varepsilon_4$, *then*

$$\varepsilon_1 \diamond \varepsilon_2 \leq \varepsilon_3 \diamond \varepsilon_4;$$

(5) $\diamond$ *is continuous on* $[0,1] \times [0,1]$.

Example 2.2. *Some standard examples of continuous t-conorms are:*

(1) *Maximum t-conorm:*

$$\varepsilon_1 \diamond \varepsilon_2 = \max\{\varepsilon_1, \varepsilon_2\}.$$

(2) *Probabilistic sum:*

$$\varepsilon_1 \diamond \varepsilon_2 = \varepsilon_1 + \varepsilon_2 - \varepsilon_1\varepsilon_2.$$

(3) *Lukasiewicz t-conorm:*

$$\varepsilon_1 \diamond \varepsilon_2 = \min\{\varepsilon_1 + \varepsilon_2, 1\}.$$

Example 2.3. *Some Fermatean-type examples of continuous t-conorms are:*

$$\varepsilon_1 \diamond \varepsilon_2 = \left[\min\{\varepsilon_1^3 + \varepsilon_2^3, 1\}\right]^{1/3},$$

$$\varepsilon_1 \diamond \varepsilon_2 = \left[\max\{\varepsilon_1^3, \varepsilon_2^3\}\right]^{1/3} = \max\{\varepsilon_1, \varepsilon_2\},$$

and

$$\varepsilon_1 \diamond \varepsilon_2 = \left(\varepsilon_1^3 + \varepsilon_2^3 - \varepsilon_1^3\varepsilon_2^3\right)^{1/3},$$

where $\varepsilon_1, \varepsilon_2 \in [0, 1]$.

Definition 2.3. *Let* X *be a nonempty set. A fuzzy set* A *on* X *is defined by a function*

$$\mu_A : X \to [0, 1],$$

where $\mu_A(x)$ *denotes the degree of membership of* x *in* A*. The value* $\mu_A(x) = 0$ *means that* x *does not belong to* A*, while* $\mu_A(x) = 1$ *means that* x *fully belongs to* A*.*

Definition 2.4. *Let* X *be a nonempty set. An intuitionistic fuzzy set* A *on* X *is defined by*

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where

$$\mu_A : X \to [0, 1] \quad \text{and} \quad \nu_A : X \to [0, 1]$$

are the membership and non-membership functions, respectively, satisfying

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

for all $x \in X$*.*

Definition 2.5. *Let* A *be an intuitionistic fuzzy set on a nonempty set* X*. The hesitation degree of* $x \in X$ *with respect to* A *is defined by*

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x).$$

Since

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1,$$

we have

$$0 \leq \pi_A(x) \leq 1.$$

Definition 2.6. *Let* X *be a nonempty set. A Fermatean fuzzy set* A *on* X *is defined by*

$$A = \{(x, \mu_A(x), \nu_A(x)) : x \in X\},$$

where

$$\mu_A : X \to [0, 1] \quad \text{and} \quad \nu_A : X \to [0, 1]$$

are the membership and non-membership functions, respectively, satisfying

$$0 \leq \mu_A^3(x) + \nu_A^3(x) \leq 1$$

for all $x \in X$*.*

Definition 2.7. *Let A be a Fermatean fuzzy set on a nonempty set X. The Fermatean hesitation degree of $x \in X$ with respect to A is defined by*

$$\pi_A(x) = \left(1 - \mu_A^3(x) - \nu_A^3(x)\right)^{\frac{1}{3}}.$$

Since

$$0 \leq \mu_A^3(x) + \nu_A^3(x) \leq 1,$$

we have

$$0 \leq \pi_A(x) \leq 1.$$

Definition 2.8. *Let X be a real linear space with $\dim X \geq n$. A function*

$$\|\cdot, \ldots, \cdot\| : X^n \to [0, \infty)$$

is called an n-norm on X if, for all $x_1, x_2, \ldots, x_n, y \in X$ and $\alpha \in \mathbb{R}$, the following conditions hold:

(1)
$$\|x_1, x_2, \ldots, x_n\| = 0$$
if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(2)
$$\|x_1, x_2, \ldots, x_n\|$$
is invariant under any permutation of $x_1, x_2, \ldots, x_n$;

(3)
$$\|x_1, x_2, \ldots, \alpha x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|;$$

(4)
$$\|x_1, x_2, \ldots, x_{n-1}, x_n + y\| \leq \|x_1, x_2, \ldots, x_{n-1}, x_n\| + \|x_1, x_2, \ldots, x_{n-1}, y\|.$$

Then $(X, \|\cdot, \ldots, \cdot\|)$ is called an n-normed linear space.

Example 2.4. *Let $X = \mathbb{R}^n$. For $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$, define*

$$\|v_1, v_2, \ldots, v_n\| = |\det(v_1, v_2, \ldots, v_n)|.$$

Then, $(\mathbb{R}^n, \|\cdot, \ldots, \cdot\|)$ is an n-normed linear space.

Definition 2.9. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed linear space. A sequence $\langle x_k \rangle$ in X is said to converge to $x \in X$ if, for every $z_1, z_2, \ldots, z_{n-1} \in X$,*

$$\|z_1, z_2, \ldots, z_{n-1}, x_k - x\| \to 0$$

as $k \to \infty$. In this case, we write

$$x_k \to x.$$

Definition 2.10. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed linear space. A sequence $\langle x_k \rangle$ in X is called a Cauchy sequence if, for every $z_1, z_2, \ldots, z_{n-1} \in X$,*

$$\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\| \to 0$$

as $k, p \to \infty$.

Definition 2.11. *An n-normed linear space $(X, \|\cdot, \ldots, \cdot\|)$ is called complete if every Cauchy sequence in X converges to an element of X with respect to the n-norm.*

Definition 2.12. *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a function.*

(1) *The function f is called non-decreasing if $s \leq t \implies f(s) \leq f(t)$ for all $s, t \in I$.*
(2) *The function f is called non-increasing if $s \leq t \implies f(s) \geq f(t)$ for all $s, t \in I$.*

Definition 2.13. *Let X be a nonempty set and let $\tau : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of τ if $\tau(x) = x$.*

Definition 2.14. *Let X be a linear space over the field $\mathbb{K}$, and let $*$ and $\diamond$ be a continuous t-norm and a continuous t-conorm, respectively. An intuitionistic fuzzy norm on X is a pair of functions*

$$\mu, \nu : X \times (0, \infty) \to [0, 1],$$

where $\mu(x,t)$ represents the degree of membership and $\nu(x,t)$ represents the degree of non-membership of x with respect to t, satisfying the following conditions for all $x, y \in X$, $t, s > 0$, and $\alpha \in \mathbb{K}$:

(1) $\mu(x,t) + \nu(x,t) \leq 1$;
(2) $\mu(x,t) > 0$ *and* $\nu(x,t) < 1$;
(3) $\mu(x,t) = 1$ *if and only if* $x = 0$;
(4) $\nu(x,t) = 0$ *if and only if* $x = 0$;
(5) *if* $\alpha \neq 0$, *then*

$$\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right), \qquad \nu(\alpha x, t) = \nu\left(x, \frac{t}{|\alpha|}\right);$$

(6)

$$\mu(x + y, t + s) \geq \mu(x,t) * \mu(y,s);$$

(7)

$$\nu(x + y, t + s) \leq \nu(x,t) \diamond \nu(y,s);$$

(8) $\lim_{t\to\infty} \mu(x,t) = 1$ *and* $\lim_{t\to\infty} \nu(x,t) = 0$;
(9) $\lim_{t\to 0^+} \mu(x,t) = 0$ *and* $\lim_{t\to 0^+} \nu(x,t) = 1$ *for* $x \neq 0$.

Then, the tuple $(X, \mu, \nu, *, \diamond)$ is called an intuitionistic fuzzy normed linear space.

Example 2.5. *Let $(X, \|\cdot\|)$ be a normed linear space. For $x \in X$ and $t > 0$, define*

$$\mu(x,t) = \frac{t}{t + \|x\|}, \qquad \nu(x,t) = \frac{\|x\|}{2t + \|x\|}.$$

Then,

$$\mu(x,t) + \nu(x,t) \leq 1.$$

Hence, (μ, ν) defines an intuitionistic fuzzy norm on X under suitable choices of the continuous t-norm and t-conorm.

Definition 2.15. *Let X be a linear space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $*$ be a continuous t-norm and let $\diamond$ be a continuous t-conorm. A pair of functions*

$$\mu, \nu : X^n \times \mathbb{R} \to [0, 1]$$

is called an intuitionistic fuzzy n-norm on X if, for all $v_1, \ldots, v_n, w_n \in X$, $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$, and $\alpha \in \mathbb{F}$, the following conditions are satisfied:

(1)

$$\mu(v_1, \ldots, v_n, \lambda) + \nu(v_1, \ldots, v_n, \lambda) \leq 1;$$

(2) *for every* $\lambda \leq 0$,

$$\mu(v_1, \ldots, v_n, \lambda) = 0 \quad \text{and} \quad \nu(v_1, \ldots, v_n, \lambda) = 1;$$

(3) *for every* $\lambda > 0$,

$$\mu(v_1, \ldots, v_n, \lambda) = 1$$

if and only if $v_1, \ldots, v_n$ are linearly dependent;

(4) *for every* $\lambda > 0$,

$$\nu(v_1, \ldots, v_n, \lambda) = 0$$

if and only if $v_1, \ldots, v_n$ are linearly dependent;

(5) $\mu(v_1, \ldots, v_n, \lambda)$ *is invariant under any permutation of* $v_1, \ldots, v_n$;
(6) $\nu(v_1, \ldots, v_n, \lambda)$ *is invariant under any permutation of* $v_1, \ldots, v_n$;

(7) *if* $\alpha \neq 0$, *then*

$$\mu(v_1, \ldots, \alpha v_n, \lambda) = \mu\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right);$$

(8) *if* $\alpha \neq 0$, *then*

$$\nu(v_1, \ldots, \alpha v_n, \lambda) = \nu\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right);$$

(9)

$$\mu(v_1, \ldots, v_{n-1}, v_n + w_n, \lambda_1 + \lambda_2) \geq \mu(v_1, \ldots, v_{n-1}, v_n, \lambda_1) * \mu(v_1, \ldots, v_{n-1}, w_n, \lambda_2);$$

(10)

$$\nu(v_1, \ldots, v_{n-1}, v_n + w_n, \lambda_1 + \lambda_2) \leq \nu(v_1, \ldots, v_{n-1}, v_n, \lambda_1) \diamond \nu(v_1, \ldots, v_{n-1}, w_n, \lambda_2);$$

(11) $\mu(v_1, \ldots, v_n, \lambda)$ *is a non-decreasing function of* $\lambda \in \mathbb{R}$ *and*

$$\lim_{\lambda \to \infty} \mu(v_1, \ldots, v_n, \lambda) = 1;$$

(12) $\nu(v_1, \ldots, v_n, \lambda)$ *is a non-increasing function of* $\lambda \in \mathbb{R}$ *and*

$$\lim_{\lambda \to \infty} \nu(v_1, \ldots, v_n, \lambda) = 0.$$

Then, the tuple $(X, \mu, \nu, *, \diamond)$ *is called an intuitionistic fuzzy n-normed linear space.*

Example 2.6. *Let* $(X, \|\cdot, \ldots, \cdot\|)$ *be an n-normed linear space. For* $v_1, \ldots, v_n \in X$ *and* $\lambda > 0$, *define*

$$\mu(v_1, \ldots, v_n, \lambda) = \frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|}$$

and

$$\nu(v_1, \ldots, v_n, \lambda) = \frac{\|v_1, \ldots, v_n\|}{2\lambda + \|v_1, \ldots, v_n\|}.$$

For $\lambda \leq 0$, *define*

$$\mu(v_1, \ldots, v_n, \lambda) = 0, \qquad \nu(v_1, \ldots, v_n, \lambda) = 1.$$

Then,

$$\mu(v_1, \ldots, v_n, \lambda) + \nu(v_1, \ldots, v_n, \lambda) \leq 1.$$

Hence, $(X, \mu, \nu, *, \diamond)$ *is an intuitionistic fuzzy n-normed linear space under suitable continuous t-norm and t-conorm.*

3. Fermatean Intuitionistic Fuzzy n-Normed Linear Space (FIF-n-NLS)

The theory of fuzzy sets was initiated by Zadeh [1], who introduced a mathematical framework for describing uncertainty through degrees of membership. Later, Atanassov [6] extended this idea by introducing intuitionistic fuzzy sets, where each element is characterized by both a membership degree and a non-membership degree. This extension provides a more flexible way to represent uncertainty, especially when the available information contains hesitation or incomplete knowledge.

In functional analysis, the concept of n-normed linear spaces provides a natural generalization of normed and 2-normed spaces. The study of n-normed spaces was developed by several authors, including Gähler and later Gunawan and Mashadi [39]. In this setting, the distance or size of an element is replaced by a function depending on n vectors, which allows one to describe linear dependence and geometric volume-like quantities in a linear space.

The interaction between fuzzy structures and normed linear spaces has led to several important generalizations. Saadati and Park [29] studied intuitionistic fuzzy normed spaces, while Vijayabalaji, Thillaigovindan, and Jun [42] introduced intuitionistic fuzzy n-normed linear

spaces and investigated convergence, Cauchy sequences, and related properties in this framework. More recently, Fermatean fuzzy sets were introduced by Senapati and Yager [21] as a wider orthopair fuzzy model satisfying the cubic condition

$$\mu^3 + \nu^3 \leq 1.$$

This condition is weaker than the intuitionistic fuzzy restriction

$$\mu + \nu \leq 1,$$

and hence it allows a broader range of admissible membership and non-membership degrees.

Motivated by these developments, we introduce the notion of a Fermatean intuitionistic fuzzy n-normed linear space. The proposed structure combines the geometric character of n-normed linear spaces with the cubic membership–non-membership condition of Fermatean fuzzy sets. In this section, we present the formal definition of this space, define the associated Fermatean hesitation degree, and discuss some fundamental concepts such as open balls, convergence, Cauchy sequences, completeness, contraction mappings, and fixed point results.

Definition 3.1. *Let X be a linear space over the field $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $*$ be a continuous t-norm and let $\diamond$ be a continuous t-conorm. Let $\widetilde{V} = (v_1, \ldots, v_n) \in X^n$ and let $\mu_F, \nu_F : X^n \times (0, \infty) \to [0, 1]$ be two functions. The tuple $\left(X^n, \mu_F, \nu_F, *, \diamond\right)$ is called a Fermatean intuitionistic fuzzy n-normed linear space, briefly an FIF-n-NLS, if the following conditions are satisfied for all $\widetilde{V} = (v_1, \ldots, v_n) \in X^n$, $\lambda, \lambda_1, \lambda_2 > 0$, $\alpha \in \mathbb{F}$, and $\widetilde{v}_n \in X$:*

(1)
$$0 < \mu_F(\widetilde{V}, \lambda) \leq 1, \qquad 0 \leq \nu_F(\widetilde{V}, \lambda) < 1.$$

(2)
$$\mu_F^3(\widetilde{V}, \lambda) + \nu_F^3(\widetilde{V}, \lambda) \leq 1.$$

(3)
$$\mu_F(\widetilde{V}, \lambda) = 1$$
if and only if $v_1, \ldots, v_n$ are linearly dependent.

(4)
$$\nu_F(\widetilde{V}, \lambda) = 0$$
if and only if $v_1, \ldots, v_n$ are linearly dependent.

(5) *$\mu_F(\widetilde{V}, \lambda)$ is invariant under any permutation of $v_1, \ldots, v_n$.*

(6) *$\nu_F(\widetilde{V}, \lambda)$ is invariant under any permutation of $v_1, \ldots, v_n$.*

(7) *If $\alpha \neq 0$, then*
$$\mu_F(v_1, \ldots, \alpha v_n, \lambda) = \mu_F\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right).$$

(8) *If $\alpha \neq 0$, then*
$$\nu_F(v_1, \ldots, \alpha v_n, \lambda) = \nu_F\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right).$$

(9)
$$\mu_F^3(v_1, \ldots, v_{n-1}, v_n, \lambda_1) * \mu_F^3(v_1, \ldots, v_{n-1}, \widetilde{v}_n, \lambda_2) \leq \mu_F^3(v_1, \ldots, v_{n-1}, v_n + \widetilde{v}_n, \lambda_1 + \lambda_2).$$

(10)
$$\nu_F^3(v_1, \ldots, v_{n-1}, v_n, \lambda_1) \diamond \nu_F^3(v_1, \ldots, v_{n-1}, \widetilde{v}_n, \lambda_2) \geq \nu_F^3(v_1, \ldots, v_{n-1}, v_n + \widetilde{v}_n, \lambda_1 + \lambda_2).$$

(11) *For every fixed $\widetilde{V} \in X^n$,*
$$\mu_F(\widetilde{V}, \cdot) : (0, \infty) \to [0, 1]$$
is continuous.

(12) *For every fixed* $\widetilde{V} \in X^n$,

$$\nu_F(\widetilde{V}, \cdot) : (0, \infty) \to [0, 1]$$

is continuous.

$$\pi_F(\widetilde{V}, \lambda) = \Big(1 - \mu_F^3(\widetilde{V}, \lambda) - \nu_F^3(\widetilde{V}, \lambda)\Big)^{1/3}.$$

Then,

$$0 \leq \pi_F(\widetilde{V}, \lambda) \leq 1.$$

Example 3.1. *Let* $X = \mathbb{R}^n$ *and define the usual* n*-norm on* X *by*

$$\|v_1, \ldots, v_n\| = |\det(v_1, \ldots, v_n)|,$$

where $v_1, \ldots, v_n \in \mathbb{R}^n$.

Let

$$a * b = ab$$

be the product t*-norm, and let*

$$a \diamond b = a + b - ab$$

be the probabilistic sum t*-conorm, for all* $a, b \in [0, 1]$.

For $\widetilde{V} = (v_1, \ldots, v_n) \in X^n$ *and* $\lambda > 0$, *define*

$$\mu_F(\widetilde{V}, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}}$$

and

$$\nu_F(\widetilde{V}, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}}.$$

Then,

$$\big(X^n, \mu_F, \nu_F, *, \diamond\big)$$

is a Fermatean intuitionistic fuzzy n*-normed linear space.*

Proof. Let

$$N(\widetilde{V}) = \|v_1, \ldots, v_n\|.$$

Then, for every $\lambda > 0$, we have

$$0 < \frac{\lambda}{\lambda + N(\widetilde{V})} \leq 1$$

and

$$0 \leq \frac{N(\widetilde{V})}{\lambda + N(\widetilde{V})} < 1.$$

Hence,

$$0 < \mu_F(\widetilde{V}, \lambda) \leq 1 \quad \text{and} \quad 0 \leq \nu_F(\widetilde{V}, \lambda) < 1.$$

Moreover,

$$\mu_F^3(\widetilde{V}, \lambda) = \frac{\lambda}{\lambda + N(\widetilde{V})}$$

and

$$\nu_F^3(\widetilde{V}, \lambda) = \frac{N(\widetilde{V})}{\lambda + N(\widetilde{V})}.$$

Therefore,

$$\mu_F^3(\widetilde{V}, \lambda) + \nu_F^3(\widetilde{V}, \lambda) = \frac{\lambda}{\lambda + N(\widetilde{V})} + \frac{N(\widetilde{V})}{\lambda + N(\widetilde{V})} = 1.$$

Thus, the Fermatean condition is satisfied.

Next, we have

$$\mu_F(\widetilde{V}, \lambda) = 1$$

if and only if

$$\frac{\lambda}{\lambda + N(\widetilde{V})} = 1,$$

which holds if and only if

$$N(\widetilde{V}) = 0.$$

Since

$$N(\widetilde{V}) = \|v_1, \ldots, v_n\| = 0$$

if and only if $v_1, \ldots, v_n$ are linearly dependent, it follows that

$$\mu_F(\widetilde{V}, \lambda) = 1$$

if and only if $v_1, \ldots, v_n$ are linearly dependent.

Similarly,

$$\nu_F(\widetilde{V}, \lambda) = 0$$

if and only if

$$N(\widetilde{V}) = 0,$$

which is equivalent to the linear dependence of $v_1, \ldots, v_n$.

Since the ordinary n-norm is invariant under any permutation of its variables, both μ_F and ν_F are invariant under any permutation of $v_1, \ldots, v_n$.

Now, let $\alpha \neq 0$. Since

$$\|v_1, \ldots, \alpha v_n\| = |\alpha| \|v_1, \ldots, v_n\|,$$

we obtain

$$\mu_F(v_1, \ldots, \alpha v_n, \lambda) = \left(\frac{\lambda}{\lambda + |\alpha| N(\widetilde{V})} \right)^{\frac{1}{3}}.$$

On the other hand,

$$\mu_F\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right) = \left(\frac{\frac{\lambda}{|\alpha|}}{\frac{\lambda}{|\alpha|} + N(\widetilde{V})} \right)^{\frac{1}{3}} = \left(\frac{\lambda}{\lambda + |\alpha| N(\widetilde{V})} \right)^{\frac{1}{3}}.$$

Thus,

$$\mu_F(v_1, \ldots, \alpha v_n, \lambda) = \mu_F\left(v_1, \ldots, v_n, \frac{\lambda}{|\alpha|}\right).$$

The proof for ν_F is similar.

It remains to verify the triangle-type conditions. Let

$$A = \|v_1, \ldots, v_{n-1}, v_n\|,$$

$$B = \|v_1, \ldots, v_{n-1}, \widetilde{v}_n\|,$$

and

$$C = \|v_1, \ldots, v_{n-1}, v_n + \widetilde{v}_n\|.$$

By the property of the ordinary n-norm, we have

$$C \leq A + B.$$

For the membership function, we need to show that

$$\mu_F^3(v_1, \ldots, v_{n-1}, v_n, \lambda_1) * \mu_F^3(v_1, \ldots, v_{n-1}, \widetilde{v}_n, \lambda_2) \leq \mu_F^3(v_1, \ldots, v_{n-1}, v_n + \widetilde{v}_n, \lambda_1 + \lambda_2).$$

Using $a * b = ab$, the left-hand side becomes

$$\frac{\lambda_1}{\lambda_1 + A} \cdot \frac{\lambda_2}{\lambda_2 + B}.$$

Since $C \leq A + B$, we have

$$\frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + C} \geq \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + A + B}.$$

Also,

$$\frac{\lambda_1}{\lambda_1 + A} \cdot \frac{\lambda_2}{\lambda_2 + B} \leq \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + A + B}.$$

Therefore,

$$\frac{\lambda_1}{\lambda_1 + A} \cdot \frac{\lambda_2}{\lambda_2 + B} \leq \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + C}.$$

Hence, the membership condition holds.

For the non-membership function, using

$$a \diamond b = a + b - ab,$$

we obtain

$$\begin{aligned} &\nu_F^3(v_1, \ldots, v_{n-1}, v_n, \lambda_1) \diamond \nu_F^3(v_1, \ldots, v_{n-1}, \widetilde{v}_n, \lambda_2) \\ &= \frac{A}{\lambda_1 + A} + \frac{B}{\lambda_2 + B} - \frac{A}{\lambda_1 + A} \cdot \frac{B}{\lambda_2 + B}. \end{aligned}$$

Equivalently,

$$\nu_F^3(v_1, \ldots, v_{n-1}, v_n, \lambda_1) \diamond \nu_F^3(v_1, \ldots, v_{n-1}, \widetilde{v}_n, \lambda_2) = 1 - \frac{\lambda_1}{\lambda_1 + A} \cdot \frac{\lambda_2}{\lambda_2 + B}.$$

From the previous inequality, we have

$$\frac{\lambda_1}{\lambda_1 + A} \cdot \frac{\lambda_2}{\lambda_2 + B} \leq \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + C}.$$

Thus,

$$1 - \frac{\lambda_1}{\lambda_1 + A} \cdot \frac{\lambda_2}{\lambda_2 + B} \geq 1 - \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 + C}.$$

Hence,

$$\nu_F^3(v_1, \ldots, v_{n-1}, v_n, \lambda_1) \diamond \nu_F^3(v_1, \ldots, v_{n-1}, \widetilde{v}_n, \lambda_2) \geq \frac{C}{\lambda_1 + \lambda_2 + C}.$$

Therefore,

$$\nu_F^3(v_1, \ldots, v_{n-1}, v_n, \lambda_1) \diamond \nu_F^3(v_1, \ldots, v_{n-1}, \widetilde{v}_n, \lambda_2) \geq \nu_F^3(v_1, \ldots, v_{n-1}, v_n + \widetilde{v}_n, \lambda_1 + \lambda_2).$$

Finally, since μ_F and ν_F are rational functions of $\lambda > 0$, both functions are continuous on $(0, \infty)$.

Therefore, $\left(X^n, \mu_F, \nu_F, *, \diamond\right)$ is a Fermatean intuitionistic fuzzy n-normed linear space. □

Definition 3.2. *Let $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ be a Fermatean intuitionistic fuzzy n-normed linear space, where*

$$\widetilde{V} = (v_1, v_2, \ldots, v_n) \in X^n$$

and $\lambda \in (0, \infty)$.

(i) *An open ball in $\mathcal{A}$ with center $\widetilde{\alpha} \in X^n$ and radius $r > 0$ is defined by*

$$B(\widetilde{\alpha}, r) = \left\{ \widetilde{\beta} \in X^n : \|\widetilde{\alpha} - \widetilde{\beta}\|_{FIF} < r \right\}.$$

(ii) *Equivalently, for* $\widetilde{V} \in X^n$, $0 < r < 1$, *and* $\lambda > 0$, *an open ball can be defined by*

$$B(\widetilde{V}, r, \lambda) = \left\{ \widetilde{W} \in X^n : \mu_F^3(\widetilde{V} - \widetilde{W}, \lambda) > 1 - r \text{ and } \nu_F^3(\widetilde{V} - \widetilde{W}, \lambda) < r \right\},$$

where

$$\widetilde{V} - \widetilde{W} = (v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n).$$

Here, $\mu_F^3(\widetilde{V} - \widetilde{W}, \lambda) > 1 - r$ means that $\widetilde{W}$ is sufficiently close to $\widetilde{V}$ with respect to the Fermatean membership degree, while $\nu_F^3(\widetilde{V} - \widetilde{W}, \lambda) < r$ means that the non-membership degree of closeness is sufficiently small.

Example 3.2. *Let* $X = \mathbb{R}^3$ *and define the usual* 3*-norm on* X *by*

$$\|u, v, w\| = |\det(u, v, w)|,$$

where $u, v, w \in \mathbb{R}^3$.

For $u, v, w \in X$ *and* $\lambda > 0$, *define*

$$\mu_F(u, v, w, \lambda) = \left(\frac{\lambda}{\lambda + \|u, v, w\|} \right)^{\frac{1}{3}}$$

and

$$\nu_F(u, v, w, \lambda) = \left(\frac{\|u, v, w\|}{\lambda + \|u, v, w\|} \right)^{\frac{1}{3}}.$$

Let

$$a * b = ab \qquad \text{and} \qquad a \diamond b = a + b - ab.$$

Then

$$\mathcal{A} = \left(X^3, \mu_F, \nu_F, *, \diamond\right)$$

is a Fermatean intuitionistic fuzzy 3*-normed linear space.*

Now take

$$\widetilde{V} = (v_1, v_2, v_3) \in X^3,$$

where

$$v_1 = (1, 0, 0), \qquad v_2 = (0, 1, 0), \qquad v_3 = (0, 0, 1).$$

Let

$$\lambda = 2 \qquad \text{and} \qquad r = \frac{1}{4}.$$

Then the open ball

$$B\left(\widetilde{V}, \frac{1}{4}, 2\right)$$

is defined by

$$B\left(\widetilde{V}, \frac{1}{4}, 2\right) = \left\{ \widetilde{W} \in X^3 : \mu_F^3(\widetilde{V} - \widetilde{W}, 2) > \frac{3}{4} \text{ and } \nu_F^3(\widetilde{V} - \widetilde{W}, 2) < \frac{1}{4} \right\}.$$

Definition 3.3. *Let* $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ *be a Fermatean intuitionistic fuzzy* n*-normed linear space. A sequence* $\langle x_k \rangle$ *in* X *is said to converge to* $x \in X$ *with respect to* $\mathcal{A}$ *if*

$$\|x_k - x\|_{FIF} \to 0 \quad \text{as} \quad k \to \infty.$$

Equivalently, $\langle x_k \rangle$ *converges to* x *if, for every* $\varepsilon \in (0, 1)$ *and every* $\lambda > 0$, *there exists* $k_0 \in \mathbb{N}$ *such that*

$$\mu_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda) > (1 - \varepsilon)^{\frac{1}{3}}$$

and

$$\nu_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon^{\frac{1}{3}}$$

for all $k \geq k_0$ *and for all* $z_1, \ldots, z_{n-1} \in X$.

In this case, we write

$$x_k \longrightarrow x \quad \text{in } \mathcal{A},$$

or

$$\lim_{k\to\infty} x_k = x.$$

Since this is a Fermatean structure, the same condition may also be written in cubic form as:

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon.$$

Example 3.3. *Let $X = \mathbb{R}^n$ and define the usual n-norm on X by*

$$\|v_1, \ldots, v_n\| = |\det(v_1, \ldots, v_n)|,$$

where $v_1, \ldots, v_n \in \mathbb{R}^n$.

Let

$$a * b = ab$$

be the product t-norm and let

$$a \diamond b = a + b - ab$$

be the probabilistic sum t-conorm, for all $a, b \in [0, 1]$.

For $\widetilde{V} = (v_1, \ldots, v_n) \in X^n$ and $\lambda > 0$, define

$$\mu_F(\widetilde{V}, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}}$$

and

$$\nu_F(\widetilde{V}, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}}.$$

Then,

$$\left(X^n, \mu_F, \nu_F, *, \diamond\right)$$

is a Fermatean intuitionistic fuzzy n-normed linear space.

Now fix a vector $u \in \mathbb{R}^n$ and define a sequence $\langle x_k \rangle$ in X by

$$x_k = \frac{u}{k}, \qquad k \in \mathbb{N}.$$

We claim that

$$x_k \to 0$$

in the Fermatean intuitionistic fuzzy n-normed linear space.

Proof. Let $\varepsilon \in (0, 1)$ and $\lambda > 0$ be given. Let $z_1, \ldots, z_{n-1} \in X$ be fixed. Since

$$x_k = \frac{u}{k},$$

we have

$$\|z_1, \ldots, z_{n-1}, x_k - 0\| = \left\|z_1, \ldots, z_{n-1}, \frac{u}{k}\right\|.$$

By the homogeneity of the ordinary n-norm,

$$\left\|z_1, \ldots, z_{n-1}, \frac{u}{k}\right\| = \frac{1}{k}\|z_1, \ldots, z_{n-1}, u\|.$$

Put

$$C = \|z_1, \ldots, z_{n-1}, u\|.$$

Then,

$$\|z_1, \ldots, z_{n-1}, x_k\| = \frac{C}{k}.$$

Therefore,

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) = \frac{\lambda}{\lambda + \frac{C}{k}} = \frac{k\lambda}{k\lambda + C},$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) = \frac{\frac{C}{k}}{\lambda + \frac{C}{k}} = \frac{C}{k\lambda + C}.$$

First, we prove the membership condition. We need

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) > 1 - \varepsilon.$$

Since

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) = \frac{k\lambda}{k\lambda + C},$$

we require

$$\frac{k\lambda}{k\lambda + C} > 1 - \varepsilon.$$

This is equivalent to

$$k\lambda > (1 - \varepsilon)(k\lambda + C).$$

Hence,

$$\varepsilon k\lambda > (1 - \varepsilon)C.$$

Thus, it is enough to choose k such that

$$k > \frac{(1 - \varepsilon)C}{\varepsilon\lambda}.$$

Next, we prove the non-membership condition. We need

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) < \varepsilon.$$

Since

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) = \frac{C}{k\lambda + C},$$

we require

$$\frac{C}{k\lambda + C} < \varepsilon.$$

This is equivalent to

$$C < \varepsilon(k\lambda + C).$$

Hence,

$$(1 - \varepsilon)C < \varepsilon k\lambda.$$

Thus it is enough to choose k such that

$$k > \frac{(1 - \varepsilon)C}{\varepsilon\lambda}.$$

Choose $k_0 \in \mathbb{N}$ such that

$$k_0 > \frac{(1 - \varepsilon)C}{\varepsilon\lambda}.$$

Then, for all $k \geq k_0$, we have

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k, \lambda) < \varepsilon.$$

Therefore,

$$x_k \to 0$$

in the Fermatean intuitionistic fuzzy n-normed linear space. □

Definition 3.4. *Let $\mathcal{A} = \big(X^n, \mu_F, \nu_F, *, \diamond\big)$ be a Fermatean intuitionistic fuzzy n-normed linear space. A sequence $\langle x_k \rangle$ in X is said to converge to $x \in X$ if, for every $\varepsilon \in (0,1)$, every $\lambda > 0$, and every $z_1, z_2, \ldots, z_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that*

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon$$

for all $k \geq k_0$.

In this case, we say that $\langle x_k \rangle$ is convergent to x in $\mathcal{A}$ and write

$$x_k \to x \quad as \quad k \to \infty.$$

Equivalently, we write

$$\lim_{k\to\infty} x_k = x.$$

Equivalently,

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) > (1 - \varepsilon)^{\frac{1}{3}}$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon^{\frac{1}{3}}.$$

Theorem 3.1. *Every intuitionistic fuzzy normed linear space naturally determines a Fermatean fuzzy normed linear space.*

Proof. Let $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed linear space. Then, for every $x \in X$ and $t > 0$, we have

$$\mu(x,t) + \nu(x,t) \leq 1.$$

Since $0 \leq \mu(x,t), \nu(x,t) \leq 1$, it follows that

$$\mu^3(x,t) \leq \mu(x,t), \qquad \nu^3(x,t) \leq \nu(x,t).$$

Therefore,

$$\mu^3(x,t) + \nu^3(x,t) \leq \mu(x,t) + \nu(x,t) \leq 1.$$

Thus the Fermatean admissibility condition is satisfied. Hence every intuitionistic fuzzy normed linear space can be regarded as a Fermatean fuzzy normed linear space whenever the remaining norm axioms are compatible. □

Theorem 3.2. *Let $\mathcal{A} = \big(X^n, \mu_F, \nu_F, *, \diamond\big)$ be a Fermatean intuitionistic fuzzy n-normed linear space as in Definition 3.1. If a sequence $\langle x_k \rangle$ in X converges to $x \in X$ and also converges to $y \in X$ in the sense of Definition 3.4, then*

$$x = y.$$

Therefore, the limit of a convergent sequence in a Fermatean intuitionistic fuzzy n-normed linear space is unique.

Proof. Suppose that the sequence $\langle x_k \rangle$ converges to both x and y in the Fermatean intuitionistic fuzzy n-normed linear space

$$\mathcal{A} = \big(X^n, \mu_F, \nu_F, *, \diamond\big).$$

That is,

$$x_k \to x \qquad \text{and} \qquad x_k \to y$$

as $k \to \infty$.

We prove that $x = y$. Assume, to the contrary, that

$$x \neq y.$$

Then,

$$x - y \neq 0.$$

Since X is an n-normed linear space, there exist $z_1, z_2, \ldots, z_{n-1} \in X$ such that

$$z_1, z_2, \ldots, z_{n-1}, x - y$$

are linearly independent.

Let $\lambda > 0$ and let $\varepsilon \in (0,1)$ be arbitrary. Since $*$ is a continuous t-norm and

$$1 * 1 = 1,$$

there exists $\delta_1 \in (0,1)$ such that

$$(1-\delta_1) * (1-\delta_1) > 1 - \varepsilon.$$

Similarly, since $\diamond$ is a continuous t-conorm and

$$0 \diamond 0 = 0,$$

there exists $\delta_2 \in (0,1)$ such that

$$\delta_2 \diamond \delta_2 < \varepsilon.$$

Choose

$$\delta = \min\{\delta_1, \delta_2\}.$$

Then,

$$(1-\delta) * (1-\delta) > 1 - \varepsilon$$

and

$$\delta \diamond \delta < \varepsilon.$$

Since $x_k \to x$, by Definition 3.4, there exists $k_1 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda/2) > 1 - \delta$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda/2) < \delta$$

for all $k \geq k_1$.

Similarly, since $x_k \to y$, there exists $k_2 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - y, \lambda/2) > 1 - \delta$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - y, \lambda/2) < \delta$$

for all $k \geq k_2$.

Let

$$k_0 = \max\{k_1, k_2\}.$$

Then, for every $k \geq k_0$, all the above inequalities hold.

Now, we observe that

$$x - y = (x - x_k) + (x_k - y).$$

By the homogeneity condition in Definition 3.1, with $\alpha = -1$, we have

$$\mu_F(z_1, \ldots, z_{n-1}, x - x_k, \lambda/2) = \mu_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda/2),$$

and

$$\nu_F(z_1, \ldots, z_{n-1}, x - x_k, \lambda/2) = \nu_F(z_1, \ldots, z_{n-1}, x_k - x, \lambda/2).$$

Using the triangle-type condition for μ_F in Definition 3.1, we obtain

$$\begin{aligned}
&\mu_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda) \\
&= \mu_F^3\big(z_1, \ldots, z_{n-1}, (x - x_k) + (x_k - y), \lambda/2 + \lambda/2\big) \\
&\geq \mu_F^3(z_1, \ldots, z_{n-1}, x - x_k, \lambda/2) * \mu_F^3(z_1, \ldots, z_{n-1}, x_k - y, \lambda/2).
\end{aligned}$$

Therefore, for every $k \geq k_0$,

$$\mu_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda) > (1-\delta) * (1-\delta).$$

By the choice of δ, we have

$$(1-\delta) * (1-\delta) > 1-\varepsilon.$$

Hence

$$\mu_F^3(z_1, \ldots, z_{n-1}, x-y, \lambda) > 1-\varepsilon.$$

Since $\varepsilon \in (0,1)$ is arbitrary, it follows that

$$\mu_F^3(z_1, \ldots, z_{n-1}, x-y, \lambda) = 1.$$

Thus

$$\mu_F(z_1, \ldots, z_{n-1}, x-y, \lambda) = 1.$$

By Definition 3.1, this implies that

$$z_1, z_2, \ldots, z_{n-1}, x-y$$

are linearly dependent.

On the other hand, by the choice of $z_1, z_2, \ldots, z_{n-1}$, the vectors

$$z_1, z_2, \ldots, z_{n-1}, x-y$$

are linearly independent. This is a contradiction.

For completeness, we may also use the non-membership function. By the triangle-type condition for ν_F in Definition 3.1, we get

$$\begin{aligned} &\nu_F^3(z_1, \ldots, z_{n-1}, x-y, \lambda) \\ &= \nu_F^3\big(z_1, \ldots, z_{n-1}, (x-x_k)+(x_k-y), \lambda/2+\lambda/2\big) \\ &\le \nu_F^3(z_1, \ldots, z_{n-1}, x-x_k, \lambda/2) \diamond \nu_F^3(z_1, \ldots, z_{n-1}, x_k-y, \lambda/2). \end{aligned}$$

Hence, for every $k \ge k_0$,

$$\nu_F^3(z_1, \ldots, z_{n-1}, x-y, \lambda) < \delta \diamond \delta.$$

By the choice of δ, we have

$$\delta \diamond \delta < \varepsilon.$$

Therefore,

$$\nu_F^3(z_1, \ldots, z_{n-1}, x-y, \lambda) < \varepsilon.$$

Since $\varepsilon \in (0,1)$ is arbitrary, it follows that

$$\nu_F^3(z_1, \ldots, z_{n-1}, x-y, \lambda) = 0.$$

Consequently,

$$\nu_F(z_1, \ldots, z_{n-1}, x-y, \lambda) = 0.$$

Again, by Definition 3.1, this implies that

$$z_1, z_2, \ldots, z_{n-1}, x-y$$

are linearly dependent, which contradicts the choice of these vectors.

Therefore, our assumption that $x \neq y$ is false. Hence,

$$x = y.$$

Thus, the point of convergence of a sequence in a Fermatean intuitionistic fuzzy n-normed linear space is unique. □

Theorem 3.3. *Let $\mathcal{A} = \big(X^n, \mu_F, \nu_F, *, \diamond\big)$ be a Fermatean intuitionistic fuzzy n-normed linear space as in Definition 3.1. A sequence $\langle x_k \rangle$ in X converges to $x \in X$ in the sense of Definition 3.4 if and only if, for every $\lambda > 0$ and every $z_1, z_2, \ldots, z_{n-1} \in X$,*

$$\lim_{k\to\infty} \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) = 1$$

and

$$\lim_{k\to\infty} \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) = 0.$$

Proof. Suppose first that $x_k \to x$ in the Fermatean intuitionistic fuzzy n-normed linear space $\mathcal{A}$. By Definition 3.4, for every $\varepsilon \in (0,1)$, every $\lambda > 0$, and every $z_1, z_2, \ldots, z_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon$$

for all $k \geq k_0$.

Since

$$0 \leq \mu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) \leq 1,$$

we have

$$0 \leq \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) \leq 1.$$

Therefore,

$$0 \leq 1 - \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon$$

for all $k \geq k_0$. Hence,

$$\left|\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) - 1\right| < \varepsilon$$

for all $k \geq k_0$. Thus,

$$\lim_{k\to\infty} \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) = 1.$$

Similarly,

$$0 \leq \nu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) \leq 1,$$

we get

$$0 \leq \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) \leq 1.$$

From

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon,$$

it follows that

$$\left|\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) - 0\right| < \varepsilon$$

for all $k \geq k_0$. Therefore,

$$\lim_{k\to\infty} \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) = 0.$$

Conversely, suppose that, for every $\lambda > 0$ and every $z_1, z_2, \ldots, z_{n-1} \in X$,

$$\lim_{k\to\infty} \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) = 1$$

and

$$\lim_{k\to\infty} \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) = 0.$$

Let $\varepsilon \in (0,1)$ be arbitrary. Since

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) \to 1,$$

there exists $k_1 \in \mathbb{N}$ such that

$$\left|\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) - 1\right| < \varepsilon$$

for all $k \geq k_1$. Hence,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) > 1 - \varepsilon$$

for all $k \geq k_1$.

Also, since

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) \to 0,$$

there exists $k_2 \in \mathbb{N}$ such that

$$\left|\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) - 0\right| < \varepsilon$$

for all $k \geq k_2$. Therefore,

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon$$

for all $k \geq k_2$.

Now, take

$$k_0 = \max\{k_1, k_2\}.$$

Then, for all $k \geq k_0$, we have

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda) < \varepsilon.$$

Therefore, by Definition 3.4, we get

$$x_k \to x.$$

This completes the proof. □

Example 3.4. *Let $X = \mathbb{R}^2$ and define the usual 2-norm on X by*

$$\|u, v\| = \left| \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \right|,$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are elements of $\mathbb{R}^2$.

Let

$$a * b = ab$$

be the product t-norm, and let

$$a \diamond b = a + b - ab$$

be the probabilistic sum t-conorm, for all $a, b \in [0, 1]$.

For $u, v \in X$ and $\lambda > 0$, define

$$\mu_F(u, v, \lambda) = \left(\frac{\lambda}{\lambda + \|u, v\|} \right)^{\frac{1}{3}}$$

and

$$\nu_F(u, v, \lambda) = \left(\frac{\|u, v\|}{\lambda + \|u, v\|} \right)^{\frac{1}{3}}.$$

Then, $\left(X^2, \mu_F, \nu_F, *, \diamond\right)$ is a Fermatean intuitionistic fuzzy 2-normed linear space.

Now, we define a sequence $\langle x_k \rangle$ in X by

$$x_k = \left(1 + \frac{1}{k}, 2 + \frac{2}{k}\right), \qquad k \in \mathbb{N}.$$

Let

$$x = (1, 2).$$

We show that

$$x_k \to x$$

in this Fermatean intuitionistic fuzzy 2-normed linear space.

Proof. Let $z = (a, b) \in \mathbb{R}^2$ be arbitrary and let $\lambda > 0$ be fixed. Then,

$$x_k - x = \left(\frac{1}{k}, \frac{2}{k}\right).$$

Hence,

$$\|z, x_k - x\| = \left| \det \begin{pmatrix} a & \frac{1}{k} \\ b & \frac{2}{k} \end{pmatrix} \right| = \left| \frac{2a - b}{k} \right| = \frac{|2a - b|}{k}.$$

Take

$$C = |2a - b|.$$

Then,

$$\|z, x_k - x\| = \frac{C}{k}.$$

Therefore,

$$\mu_F^3(z, x_k - x, \lambda) = \frac{\lambda}{\lambda + \|z, x_k - x\|} = \frac{\lambda}{\lambda + \frac{C}{k}} = \frac{k\lambda}{k\lambda + C}.$$

Thus,

$$\lim_{k\to\infty} \mu_F^3(z, x_k - x, \lambda) = \lim_{k\to\infty} \frac{k\lambda}{k\lambda + C} = 1.$$

Similarly,

$$\nu_F^3(z, x_k - x, \lambda) = \frac{\|z, x_k - x\|}{\lambda + \|z, x_k - x\|} = \frac{\frac{C}{k}}{\lambda + \frac{C}{k}} = \frac{C}{k\lambda + C}.$$

Hence,

$$\lim_{k\to\infty} \nu_F^3(z, x_k - x, \lambda) = \lim_{k\to\infty} \frac{C}{k\lambda + C} = 0.$$

Since $z \in \mathbb{R}^2$ and $\lambda > 0$ were arbitrary, we have

$$\lim_{k\to\infty} \mu_F^3(z, x_k - x, \lambda) = 1$$

and

$$\lim_{k\to\infty} \nu_F^3(z, x_k - x, \lambda) = 0.$$

Therefore, by Theorem 3.3,

$$x_k \to x.$$

□

4. Cauchy Sequences in Fermatean Intuitionistic Fuzzy n-Normed Linear Spaces

Cauchy sequences play an important role in the study of convergence and completeness. In classical normed linear spaces, a sequence is Cauchy if the distance between its terms becomes arbitrarily small as the indices become sufficiently large. In a Fermatean intuitionistic fuzzy n-normed linear space, this idea is expressed through the behavior of the Fermatean membership and non-membership functions.

More precisely, the closeness between two terms x_k and x_p of a sequence is measured by

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda)$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda),$$

where $z_1, z_2, \ldots, z_{n-1} \in X$ and $\lambda > 0$. A sequence is expected to be Cauchy whenever the membership degree tends to 1 and the non-membership degree tends to 0 as $k, p \to \infty$. This agrees with the usual interpretation that the terms of the sequence become increasingly close to one another.

The following definition formalizes the notion of a Cauchy sequence in a Fermatean intuitionistic fuzzy n-normed linear space.

Definition 4.1. *Let $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ be a Fermatean intuitionistic fuzzy n-normed linear space. A sequence $\langle x_k \rangle$ in X is called a Cauchy sequence in $\mathcal{A}$ if, for every $\varepsilon \in (0, 1)$, every $\lambda > 0$, and every $z_1, z_2, \ldots, z_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that*

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$. Equivalently,

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > (1-\varepsilon)^{\frac{1}{3}}$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}}.$$

Example 4.1. *Let $X = \mathbb{R}^n$ and define the usual n-norm on X by*

$$\|v_1, v_2, \ldots, v_n\| = |\det(v_1, v_2, \ldots, v_n)|,$$

where $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$.

Let $a * b = ab$ be the product t-norm and let $a \diamond b = a + b - ab$ be the probabilistic sum t-conorm, for all $a, b \in [0, 1]$.

For $v_1, v_2, \ldots, v_n \in X$ and $\lambda > 0$, define

$$\mu_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, v_2, \ldots, v_n\|}\right)^{\frac{1}{3}}$$

and

$$\nu_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\|v_1, v_2, \ldots, v_n\|}{\lambda + \|v_1, v_2, \ldots, v_n\|}\right)^{\frac{1}{3}}.$$

Then, $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ is a Fermatean intuitionistic fuzzy n-normed linear space.

Now, fix $u \in X$ and define a sequence $\langle x_k \rangle$ in X by

$$x_k = \frac{u}{k}, \qquad k \in \mathbb{N}.$$

Then, $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$.

Proof. Let $\varepsilon \in (0, 1)$, $\lambda > 0$, and $z_1, z_2, \ldots, z_{n-1} \in X$ be arbitrary.

For $k, p \in \mathbb{N}$, we have

$$x_k - x_p = \frac{u}{k} - \frac{u}{p} = \left(\frac{1}{k} - \frac{1}{p}\right) u.$$

By the homogeneity property of the usual n-norm,

$$\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\| = \left|\frac{1}{k} - \frac{1}{p}\right| \|z_1, z_2, \ldots, z_{n-1}, u\|.$$

Put

$$C = \|z_1, z_2, \ldots, z_{n-1}, u\|.$$

Then,

$$\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\| = C\left|\frac{1}{k} - \frac{1}{p}\right|.$$

Since

$$\left|\frac{1}{k} - \frac{1}{p}\right| \leq \frac{1}{k} + \frac{1}{p},$$

we get, for all $k, p \geq k_0$,

$$\left|\frac{1}{k} - \frac{1}{p}\right| \leq \frac{2}{k_0}.$$

Choose $k_0 \in \mathbb{N}$ such that

$$\frac{2C}{k_0} < \frac{\varepsilon\lambda}{1-\varepsilon}.$$

Then, for all $k, p \geq k_0$,

$$\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\| < \frac{\varepsilon\lambda}{1-\varepsilon}.$$

Now,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = \frac{\lambda}{\lambda + \|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\|}.$$

Therefore, for all $k, p \geq k_0$,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > \frac{\lambda}{\lambda + \frac{\varepsilon\lambda}{1-\varepsilon}} = 1 - \varepsilon.$$

Similarly,

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = \frac{\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\|}{\lambda + \|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\|}.$$

Since

$$\|z_1, z_2, \ldots, z_{n-1}, x_k - x_p\| < \frac{\varepsilon\lambda}{1-\varepsilon},$$

we obtain

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \frac{\frac{\varepsilon\lambda}{1-\varepsilon}}{\lambda + \frac{\varepsilon\lambda}{1-\varepsilon}} = \varepsilon.$$

Hence, for every $\varepsilon \in (0, 1)$, every $\lambda > 0$, and every $z_1, z_2, \ldots, z_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$.

Equivalently,

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > (1 - \varepsilon)^{\frac{1}{3}}$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}}$$

for all $k, p \geq k_0$.

Therefore, by Definition 4.1, the sequence $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$. □

Example 4.2. *Let $X = \mathbb{R}^3$ and define the usual 3-norm on X by*

$$\|v_1, v_2, v_3\| = |\det(v_1, v_2, v_3)|,$$

where $v_1, v_2, v_3 \in \mathbb{R}^3$.

Take $a * b = ab$ be the product t-norm and let $a \diamond b = a + b - ab$ be the probabilistic sum t-conorm, for all $a, b \in [0, 1]$.

For $v_1, v_2, v_3 \in X$ and $\lambda > 0$, define

$$\mu_F(v_1, v_2, v_3, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, v_2, v_3\|}\right)^{\frac{1}{3}}$$

and

$$\nu_F(v_1, v_2, v_3, \lambda) = \left(\frac{\|v_1, v_2, v_3\|}{\lambda + \|v_1, v_2, v_3\|}\right)^{\frac{1}{3}}.$$

Then, $\mathcal{A} = \left(X^3, \mu_F, \nu_F, *, \diamond\right)$ is a Fermatean intuitionistic fuzzy 3-normed linear space.

Define a sequence $\langle x_k \rangle$ in X by

$$x_k = \left(1 + \frac{1}{k}, 2 + \frac{1}{k^2}, 3 + \frac{1}{k^3}\right), \qquad k \in \mathbb{N}.$$

Then, $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$.

Proof. Let $\varepsilon \in (0,1)$, $\lambda > 0$, and $z_1, z_2 \in X$ be arbitrary. We prove that there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, z_2, x_k - x_p, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, z_2, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$.

For $k, p \in \mathbb{N}$, we have

$$x_k - x_p = \left(\frac{1}{k} - \frac{1}{p}, \frac{1}{k^2} - \frac{1}{p^2}, \frac{1}{k^3} - \frac{1}{p^3}\right).$$

As the mapping $w \mapsto \det(z_1, z_2, w)$ is linear in w, there exists a constant $C_{z_1,z_2} \geq 0$, depending only on z_1 and z_2, such that

$$|\det(z_1, z_2, w)| \leq C_{z_1,z_2} \left(|w_1| + |w_2| + |w_3|\right)$$

for every $w = (w_1, w_2, w_3) \in \mathbb{R}^3$.

Hence,

$$\|z_1, z_2, x_k - x_p\| = |\det(z_1, z_2, x_k - x_p)|$$
$$\leq C_{z_1,z_2} \left(\left|\frac{1}{k} - \frac{1}{p}\right| + \left|\frac{1}{k^2} - \frac{1}{p^2}\right| + \left|\frac{1}{k^3} - \frac{1}{p^3}\right|\right).$$

For $k, p \geq k_0$, we have

$$\left|\frac{1}{k} - \frac{1}{p}\right| \leq \frac{2}{k_0},$$
$$\left|\frac{1}{k^2} - \frac{1}{p^2}\right| \leq \frac{2}{k_0^2} \leq \frac{2}{k_0},$$

and

$$\left|\frac{1}{k^3} - \frac{1}{p^3}\right| \leq \frac{2}{k_0^3} \leq \frac{2}{k_0}.$$

Therefore,

$$\|z_1, z_2, x_k - x_p\| \leq \frac{6C_{z_1,z_2}}{k_0}$$

for all $k, p \geq k_0$.

Choose $k_0 \in \mathbb{N}$ sufficiently large such that

$$\frac{6C_{z_1,z_2}}{k_0} < \frac{\varepsilon\lambda}{1-\varepsilon}.$$

Then, for all $k, p \geq k_0$,

$$\|z_1, z_2, x_k - x_p\| < \frac{\varepsilon\lambda}{1-\varepsilon}.$$

Now,

$$\mu_F^3(z_1, z_2, x_k - x_p, \lambda) = \frac{\lambda}{\lambda + \|z_1, z_2, x_k - x_p\|}.$$

Thus,

$$\mu_F^3(z_1, z_2, x_k - x_p, \lambda) > \frac{\lambda}{\lambda + \frac{\varepsilon\lambda}{1-\varepsilon}} = 1 - \varepsilon.$$

Similarly,

$$\nu_F^3(z_1, z_2, x_k - x_p, \lambda) = \frac{\|z_1, z_2, x_k - x_p\|}{\lambda + \|z_1, z_2, x_k - x_p\|}.$$

Since

$$\|z_1, z_2, x_k - x_p\| < \frac{\varepsilon\lambda}{1-\varepsilon},$$

we obtain

$$\nu_F^3(z_1, z_2, x_k - x_p, \lambda) < \frac{\frac{\varepsilon\lambda}{1-\varepsilon}}{\lambda + \frac{\varepsilon\lambda}{1-\varepsilon}} = \varepsilon.$$

Hence,

$$\mu_F^3(z_1, z_2, x_k - x_p, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, z_2, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$.

Equivalently,

$$\mu_F(z_1, z_2, x_k - x_p, \lambda) > (1 - \varepsilon)^{\frac{1}{3}}$$

and

$$\nu_F(z_1, z_2, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}}$$

for all $k, p \geq k_0$.

Therefore, by Definition 4.1, $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$. □

Theorem 4.1. *Let $\mathcal{A} = \big(X^n, \mu_F, \nu_F, *, \diamond\big)$ be a Fermatean intuitionistic fuzzy n-normed linear space as in Definition 3.1. If a sequence $\langle x_k \rangle$ in X converges to $x \in X$ in the sense of Definition 3.4, then $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$ in the sense of Definition 4.1.*

Proof. Suppose that

$$x_k \to x$$

in the Fermatean intuitionistic fuzzy n-normed linear space

$$\mathcal{A} = \big(X^n, \mu_F, \nu_F, *, \diamond\big).$$

We prove that $\langle x_k \rangle$ is a Cauchy sequence.

Let $\varepsilon \in (0, 1)$, $\lambda > 0$, and $z_1, z_2, \ldots, z_{n-1} \in X$ be arbitrary.

Since $*$ is a continuous t-norm and

$$1 * 1 = 1,$$

there exists $\delta_1 \in (0, 1)$ such that

$$(1 - \delta_1) * (1 - \delta_1) > 1 - \varepsilon.$$

Also, since $\diamond$ is a continuous t-conorm and

$$0 \diamond 0 = 0,$$

there exists $\delta_2 \in (0, 1)$ such that

$$\delta_2 \diamond \delta_2 < \varepsilon.$$

Let

$$\delta = \min\{\delta_1, \delta_2\}.$$

Then,

$$(1 - \delta) * (1 - \delta) > 1 - \varepsilon$$

and

$$\delta \diamond \delta < \varepsilon.$$

Since $x_k \to x$, by Definition 3.4, there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda/2) > 1 - \delta$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda/2) < \delta$$

for all $k \geq k_0$.

Now, let $k, p \geq k_0$. We have

$$x_k - x_p = (x_k - x) + (x - x_p).$$

By the homogeneity property in Definition 3.1, with $\alpha = -1$, we get

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) = \mu_F(z_1, z_2, \ldots, z_{n-1}, x_p - x, \lambda/2),$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) = \nu_F(z_1, z_2, \ldots, z_{n-1}, x_p - x, \lambda/2).$$

Hence,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) > 1 - \delta$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2) < \delta$$

for all $p \geq k_0$.

Using the triangle-type condition for μ_F in Definition 3.1, we obtain

$$\begin{aligned}
&\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \\
&= \mu_F^3\big(z_1, z_2, \ldots, z_{n-1}, (x_k - x) + (x - x_p), \lambda/2 + \lambda/2\big) \\
&\geq \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda/2) * \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2).
\end{aligned}$$

Since both factors on the right-hand side are greater than $1 - \delta$, we get

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > (1 - \delta) * (1 - \delta).$$

By the choice of δ,

$$(1 - \delta) * (1 - \delta) > 1 - \varepsilon.$$

Therefore,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon.$$

Next, using the triangle-type condition for ν_F in Definition 3.1, we have

$$\begin{aligned}
&\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \\
&= \nu_F^3\big(z_1, z_2, \ldots, z_{n-1}, (x_k - x) + (x - x_p), \lambda/2 + \lambda/2\big) \\
&\leq \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x, \lambda/2) \diamond \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x - x_p, \lambda/2).
\end{aligned}$$

Since both terms on the right-hand side are less than δ, by the monotonicity of the continuous t-conorm $\diamond$, we get

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \delta \diamond \delta.$$

By the choice of δ,

$$\delta \diamond \delta < \varepsilon.$$

Thus,

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon.$$

Therefore, for every $\varepsilon \in (0, 1)$, every $\lambda > 0$, and every $z_1, z_2, \ldots, z_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$.

Equivalently,

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > (1 - \varepsilon)^{\frac{1}{3}}$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}}$$

for all $k, p \geq k_0$.

Hence, by Definition 4.1, the sequence $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$. □

5. Completeness in Fermatean Intuitionistic Fuzzy n-Normed Linear Spaces

Completeness is one of the central notions in the study of normed and fuzzy normed structures. In classical analysis, completeness guarantees that every Cauchy sequence has a limit inside the space. This property is essential for developing convergence theory, fixed point results, and approximation methods. In the framework of Fermatean intuitionistic fuzzy n-normed linear spaces, the concept of completeness is described by using the Fermatean membership and non-membership functions.

In particular, a Cauchy sequence in this setting is not measured by an ordinary distance only. Instead, it is characterized by the conditions

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \to 1$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \to 0$$

as $k, p \to \infty$, for every $\lambda > 0$ and every $z_1, z_2, \ldots, z_{n-1} \in X$. Thus, completeness means that such a sequence must converge to some element of the underlying linear space X with respect to the Fermatean intuitionistic fuzzy n-norm structure.

The following definition introduces completeness in a Fermatean intuitionistic fuzzy n-normed linear space.

Definition 5.1. *A Fermatean intuitionistic fuzzy n-normed linear space $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ is said to be complete if every Cauchy sequence in $\mathcal{A}$ converges to an element of X with respect to the Fermatean intuitionistic fuzzy n-norm structure.*

Example 5.1. *Let $X = \mathbb{R}^n$ and define the usual n-norm by*

$$\|v_1, v_2, \ldots, v_n\| = |\det(v_1, v_2, \ldots, v_n)|.$$

For $\lambda > 0$, define

$$\mu_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}}$$

and

$$\nu_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\|v_1, \ldots, v_n\|}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}}.$$

Let

$$a * b = ab \quad \text{and} \quad a \diamond b = a + b - ab.$$

Then, $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ is a complete Fermatean intuitionistic fuzzy n-normed linear space.

Proof. Let $\langle x_k \rangle$ be a Cauchy sequence in $\mathcal{A}$. Then, by Definition 4.1, for every $\varepsilon \in (0, 1)$, $\lambda > 0$, and $z_1, \ldots, z_{n-1} \in X$,

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all sufficiently large k, p.

Hence,

$$\|z_1, \ldots, z_{n-1}, x_k - x_p\| \to 0 \quad \text{as} \quad k, p \to \infty.$$

Since $X = \mathbb{R}^n$ is complete, there exists $x \in \mathbb{R}^n$ such that

$$x_k \to x.$$

Therefore,

$$\|z_1, \ldots, z_{n-1}, x_k - x\| \to 0.$$

Consequently,

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) \to 1$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) \to 0.$$

Thus $\langle x_k \rangle$ converges to x in $\mathcal{A}$. Hence, $\mathcal{A}$ is complete. □

The next lemma gives an equivalent characterization of Cauchy sequences in terms of the limiting behavior of the Fermatean membership and non-membership functions. This characterization is useful because it allows us to verify Cauchy sequences by checking the limits of μ_F^3 and ν_F^3 directly.

Lemma 5.1. *Let $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ be a Fermatean intuitionistic fuzzy n-normed linear space as in Definition 3.1. A sequence $\langle x_k \rangle$ in X is Cauchy in $\mathcal{A}$ if and only if, for every $\lambda > 0$ and every $z_1, z_2, \ldots, z_{n-1} \in X$,*

$$\lim_{k,p\to\infty} \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 1$$

and

$$\lim_{k,p\to\infty} \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 0.$$

Proof. We prove both directions.

First, suppose that $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$. By Definition 4.1, for every $\varepsilon \in (0, 1)$, every $\lambda > 0$, and every $z_1, z_2, \ldots, z_{n-1} \in X$, there exists $k_0 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$.

Since

$$0 \leq \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \leq 1,$$

we have

$$0 \leq 1 - \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$. Hence,

$$\left|\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) - 1\right| < \varepsilon$$

for all $k, p \geq k_0$. Therefore,

$$\lim_{k,p\to\infty} \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 1.$$

Similarly, since

$$0 \leq \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \leq 1,$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_0$, we get

$$\left|\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) - 0\right| < \varepsilon$$

for all $k, p \geq k_0$. Thus,

$$\lim_{k,p\to\infty} \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 0.$$

Conversely, suppose that, for every $\lambda > 0$ and every $z_1, z_2, \ldots, z_{n-1} \in X$,

$$\lim_{k,p\to\infty} \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 1$$

and

$$\lim_{k,p\to\infty} \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) = 0.$$

Let $\varepsilon \in (0,1)$ be arbitrary. Since

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \to 1 \quad \text{as} \quad k, p \to \infty,$$

there exists $k_1 \in \mathbb{N}$ such that

$$\left|\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) - 1\right| < \varepsilon$$

for all $k, p \geq k_1$. Hence,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon$$

for all $k, p \geq k_1$.

Also, since

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \to 0 \quad \text{as} \quad k, p \to \infty,$$

there exists $k_2 \in \mathbb{N}$ such that

$$\left|\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) - 0\right| < \varepsilon$$

for all $k, p \geq k_2$. Therefore,

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon$$

for all $k, p \geq k_2$.

Now, take

$$k_0 = \max\{k_1, k_2\}.$$

Then, for all $k, p \geq k_0$, we have

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon.$$

Equivalently,

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) > (1 - \varepsilon)^{\frac{1}{3}}$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) < \varepsilon^{\frac{1}{3}}$$

for all $k, p \geq k_0$.

Therefore, by Definition 4.1, the sequence $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$.

Hence, $\langle x_k \rangle$ is Cauchy in $\mathcal{A}$ if and only if

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \to 1$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x_k - x_p, \lambda) \to 0$$

as $k, p \to \infty$. □

6. Contraction Mappings and Fixed Point Theorem in FIF-n-NLS

Fixed point theory is one of the most important tools in nonlinear analysis. It is widely used to prove the existence and uniqueness of solutions of operator equations, differential equations, integral equations, and iterative processes. In classical metric and normed spaces, contraction mappings play a central role because they guarantee the convergence of successive iterations to a unique fixed point.

In the framework of Fermatean intuitionistic fuzzy n-normed linear spaces, the notion of contraction must be formulated in terms of the Fermatean membership and non-membership functions. Instead of measuring the distance between two points only by an ordinary norm, we compare the degrees

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x-y, \lambda)$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x-y, \lambda),$$

where $x, y \in X$, $z_1, z_2, \ldots, z_{n-1} \in X$, and $\lambda > 0$. A contraction mapping should increase the membership degree of closeness and decrease the non-membership degree of closeness after applying the mapping.

To obtain a more flexible contraction condition, we introduce a pair of control functions (φ, ψ). The function φ controls the membership component, while ψ controls the non-membership component. These functions are required to preserve the Fermatean cubic condition, so that the controlled membership and non-membership degrees remain admissible in the Fermatean intuitionistic fuzzy sense. The following definition introduces the Fermatean intuitionistic fuzzy cubic-control function.

Definition 6.1. *Let $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ be a Fermatean intuitionistic fuzzy n-normed linear space. A pair of functions (φ, ψ) is called a Fermatean intuitionistic fuzzy cubic-control function if $\varphi, \psi : [0,1] \to [0,1]$ satisfy the following conditions:*

(1) *φ and ψ are continuous;*

(2) *φ and ψ are non-decreasing;*

(3)
$$\varphi(t) \geq 0 \quad \textit{and} \quad \psi(t) \geq 0$$
for all $t \in [0,1]$;

(4)
$$\varphi(1) = 1 \quad \textit{and} \quad \psi(0) = 0;$$

(5) *for every $\widetilde{V} = (v_1, \ldots, v_n) \in X^n$ and $\lambda > 0$,*
$$\varphi\left(\mu_F(\widetilde{V}, \lambda)\right)^3 + \psi\left(\nu_F(\widetilde{V}, \lambda)\right)^3 \leq 1,$$
whenever
$$\mu_F^3(\widetilde{V}, \lambda) + \nu_F^3(\widetilde{V}, \lambda) \leq 1.$$

Then, the pair (φ, ψ) is called a Fermatean intuitionistic fuzzy cubic-control function on $\mathcal{A}$.

The function φ controls the membership degree, while ψ controls the non-membership degree. The condition
$$\varphi\left(\mu_F(\widetilde{V}, \lambda)\right)^3 + \psi\left(\nu_F(\widetilde{V}, \lambda)\right)^3 \leq 1$$
ensures that the controlled pair still satisfies the Fermatean cubic condition.

After introducing the cubic-control functions, we now define a Fermatean intuitionistic fuzzy contraction mapping. This type of mapping is designed to control the Fermatean membership and non-membership degrees between the images of two points under the mapping. The parameters ξ and η allow the contraction condition to include both the deviation from full membership and the influence of the control functions.

Definition 6.2. *Let $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ be a Fermatean intuitionistic fuzzy n-normed linear space, and let (φ, ψ) be a Fermatean intuitionistic fuzzy cubic-control function. Let $\xi, \eta \geq 0$ be fixed constants.*

A mapping $\tau : X \to X$ is called a Fermatean intuitionistic fuzzy (φ, ψ)-contraction mapping on $\mathcal{A}$ if, for every $x, y \in X$, every $z_1, z_2, \ldots, z_{n-1} \in X$, and every $\lambda > 0$, the following conditions hold:

$$\mu_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \geq \frac{\mu_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda)}{1 + \xi\left[1 - \mu_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda)\right] + \eta\varphi\left(\mu_F(z_1, \ldots, z_{n-1}, x - y, \lambda)\right)}$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \leq \frac{\nu_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda)}{1 + \xi\nu_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda) + \eta\psi\left(\nu_F(z_1, \ldots, z_{n-1}, x - y, \lambda)\right)}.$$

Then, τ is called a Fermatean intuitionistic fuzzy contraction mapping on $\mathcal{A}$.

Example 6.1. *Let $X = \mathbb{R}^n$ and define the usual n-norm on X by*

$$\|v_1, v_2, \ldots, v_n\| = \left|\det(v_1, v_2, \ldots, v_n)\right|,$$

where $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$.

Let

$$a * b = ab \qquad \text{and} \qquad a \diamond b = a + b - ab$$

for all $a, b \in [0, 1]$.

For $v_1, v_2, \ldots, v_n \in X$ and $\lambda > 0$, define

$$\mu_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, v_2, \ldots, v_n\|}\right)^{\frac{1}{3}}$$

and

$$\nu_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\|v_1, v_2, \ldots, v_n\|}{\lambda + \|v_1, v_2, \ldots, v_n\|}\right)^{\frac{1}{3}}.$$

Then,

$$\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$$

is a Fermatean intuitionistic fuzzy n-normed linear space.

Define

$$\varphi(t) = t^2 \qquad \text{and} \qquad \psi(t) = t^2, \qquad t \in [0, 1].$$

Then, (φ, ψ) is a Fermatean intuitionistic fuzzy cubic-control function. Take

$$\xi = 1 \qquad \text{and} \qquad \eta = 1.$$

Now, we define a mapping

$$\tau : X \to X$$

by

$$\tau(x) = 0 \qquad \text{for all } x \in X.$$

Then, τ is a Fermatean intuitionistic fuzzy (φ, ψ)-contraction mapping on $\mathcal{A}$.

Proof. Let $x, y \in X$, let $z_1, z_2, \ldots, z_{n-1} \in X$, and let $\lambda > 0$ be arbitrary. Since

$$\tau(x) = 0 \qquad \text{and} \qquad \tau(y) = 0,$$

we have

$$\tau x - \tau y = 0.$$

Therefore,

$$z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y = z_1, z_2, \ldots, z_{n-1}, 0$$

are linearly dependent. Hence, by the definition of μ_F and ν_F,

$$\mu_F(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 1$$

and

$$\nu_F(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 0.$$

Consequently,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 1$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 0.$$

Put

$$M = \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)$$

and

$$N = \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda).$$

Since $0 \leq M \leq 1$ and $0 \leq N \leq 1$, we have

$$1 + \xi(1 - M) + \eta\varphi\left(\mu_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right) \geq 1.$$

Thus,

$$\frac{M}{1 + \xi(1 - M) + \eta\varphi\left(\mu_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right)} \leq M \leq 1.$$

Therefore,

$$\begin{aligned} &\mu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 1 \\ &\geq \frac{\mu_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)}{1 + \xi\left[1 - \mu_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right] + \eta\varphi\left(\mu_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right)}. \end{aligned}$$

Similarly,

$$1 + \xi N + \eta\psi\left(\nu_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right) \geq 1.$$

Hence,

$$\frac{N}{1 + \xi N + \eta\psi\left(\nu_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right)} \geq 0.$$

Since

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) = 0,$$

we obtain

$$\nu_F^3(z_1, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \leq \frac{\nu_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda)}{1 + \xi\nu_F^3(z_1, \ldots, z_{n-1}, x - y, \lambda) + \eta\psi\left(\nu_F(z_1, \ldots, z_{n-1}, x - y, \lambda)\right)}.$$

Therefore, both contraction inequalities in Definition 6.2 are satisfied. Hence τ is a Fermatean intuitionistic fuzzy (φ, ψ)-contraction mapping on $\mathcal{A}$. □

The notion of a fixed point in this setting is the same as in classical fixed point theory: a point remains unchanged under the action of the mapping. However, the existence and uniqueness of such a point will be established using the structure of the Fermatean intuitionistic fuzzy n-normed linear space and the contraction conditions defined above.

Definition 6.3. *Let $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ be a Fermatean intuitionistic fuzzy n-normed linear space, and let $\tau : X \to X$ be a Fermatean intuitionistic fuzzy n-contraction mapping on $\mathcal{A}$.*

A point $x \in X$ is called a fixed point of τ in $\mathcal{A}$ if $\tau(x) = x$.

Example 6.2. *Let $X = \mathbb{R}^n$ and define the usual n-norm on X by*

$$\|v_1, v_2, \ldots, v_n\| = \left|\det(v_1, v_2, \ldots, v_n)\right|,$$

where $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$.

For $v_1, v_2, \ldots, v_n \in X$ and $\lambda > 0$, define

$$\mu_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, v_2, \ldots, v_n\|}\right)^{\frac{1}{3}}$$

and

$$\nu_F(v_1, v_2, \ldots, v_n, \lambda) = \left(\frac{\|v_1, v_2, \ldots, v_n\|}{\lambda + \|v_1, v_2, \ldots, v_n\|} \right)^{\frac{1}{3}}.$$

Let

$$a * b = ab \qquad \text{and} \qquad a \diamond b = a + b - ab$$

for all $a, b \in [0, 1]$. *Then,* $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$ *is a Fermatean intuitionistic fuzzy* n*-normed linear space.*

Define a mapping

$$\tau : X \to X$$

by

$$\tau(x) = c + \frac{1}{2}x,$$

where

$$c = (1, 2, 3, \ldots, n) \in \mathbb{R}^n.$$

We claim that

$$x^* = 2c = (2, 4, 6, \ldots, 2n)$$

is a fixed point of τ *in* $\mathcal{A}$.

Proof. By Definition 6.3, a point $x^* \in X$ is a fixed point of τ if $\tau(x^*) = x^*$. Take $x^* = 2c$. Then,

$$\tau(x^*) = c + \frac{1}{2}x^*.$$

Since $x^* = 2c$, we obtain

$$\tau(x^*) = c + \frac{1}{2}(2c) = c + c = 2c.$$

Thus,

$$\tau(x^*) = x^*.$$

Hence,

$$x^* = 2c = (2, 4, 6, \ldots, 2n)$$

is a fixed point of τ in the Fermatean intuitionistic fuzzy n-normed linear space $\mathcal{A} = \left(X^n, \mu_F, \nu_F, *, \diamond\right)$. Moreover, if $x \in X$ is any fixed point of τ, then $\tau(x) = x$. Therefore, $c + \frac{1}{2}x = x$. Hence, $c = \frac{1}{2}x$. Thus, $x = 2c$. Therefore, the fixed point of τ is unique. □

The following theorem is a Banach-type fixed point theorem in a complete Fermatean intuitionistic fuzzy n-normed linear space. It shows that if a mapping satisfies an appropriate Fermatean intuitionistic fuzzy contraction condition, then the Picard iteration

$$x_{k+1} = \tau x_k$$

converges to a unique fixed point of τ. This result provides the foundation for applying the proposed space to operator equations and related problems.

Theorem 6.1. *Let* $\mathcal{A} = \left(X^n, \mu_F, \nu_F, \min, \max\right)$ *be a complete Fermatean intuitionistic fuzzy* n*-normed linear space as in Definition 3.1. Suppose that*

$$\tau : X \to X$$

is a Fermatean intuitionistic fuzzy n*-contraction mapping, that is, there exists a constant* $\rho \in (0, 1)$ *such that, for every* $x, y \in X$, *every* $z_1, z_2, \ldots, z_{n-1} \in X$, *and every* $\lambda > 0$,

$$\mu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \geq \mu_F^3\left(z_1, z_2, \ldots, z_{n-1}, x - y, \frac{\lambda}{\rho}\right)$$

and

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \leq \nu_F^3\left(z_1, z_2, \ldots, z_{n-1}, x - y, \frac{\lambda}{\rho}\right).$$

Then, τ has a unique fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\langle x_k \rangle$ in X by

$$x_{k+1} = \tau x_k, \qquad k = 0, 1, 2, \ldots.$$

We first prove that $\langle x_k \rangle$ is a Cauchy sequence in $\mathcal{A}$.

Let $\varepsilon \in (0,1)$, $\lambda > 0$, and $z_1, z_2, \ldots, z_{n-1} \in X$ be arbitrary. For $j \geq 1$, since

$$x_{j+1} - x_j = \tau x_j - \tau x_{j-1},$$

the contraction condition gives

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, \lambda) \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, x_j - x_{j-1}, \frac{\lambda}{\rho}\right)$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, \lambda) \leq \nu_F^3\left(z_1, \ldots, z_{n-1}, x_j - x_{j-1}, \frac{\lambda}{\rho}\right).$$

By repeated application, for every $j \geq 1$ and every $t > 0$, we obtain

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t) \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, \frac{t}{\rho^j}\right)$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t) \leq \nu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, \frac{t}{\rho^j}\right).$$

Now, let $p > k$. Then,

$$x_p - x_k = (x_p - x_{p-1}) + (x_{p-1} - x_{p-2}) + \cdots + (x_{k+1} - x_k).$$

For $j = k, k+1, \ldots, p-1$, put

$$t_j = (1-\rho)\rho^{j-k}\lambda.$$

Then,

$$\sum_{j=k}^{p-1} t_j = (1-\rho)\lambda \sum_{j=k}^{p-1} \rho^{j-k} < \lambda.$$

Since μ_F is non-decreasing with respect to λ, and since the t-norm is chosen as min, the triangle-type condition gives

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) \geq \min_{k \leq j \leq p-1} \mu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t_j).$$

Using the estimate for successive differences, we get

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t_j) \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, \frac{t_j}{\rho^j}\right).$$

But

$$\frac{t_j}{\rho^j} = \frac{(1-\rho)\rho^{j-k}\lambda}{\rho^j} = (1-\rho)\lambda\rho^{-k}.$$

Hence,

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, (1-\rho)\lambda\rho^{-k}\right).$$

Since $\rho \in (0,1)$, we have

$$(1-\rho)\lambda\rho^{-k} \to \infty \quad \text{as} \quad k \to \infty.$$

Therefore, by the defining property of the FIF-n-norm,

$$\mu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, (1-\rho)\lambda\rho^{-k}\right) \to 1.$$

Thus, there exists $k_1 \in \mathbb{N}$ such that

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) > 1 - \varepsilon$$

for all $p > k \geq k_1$.

Similarly, since the t-conorm is chosen as max, the triangle-type condition for ν_F gives

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) \leq \max_{k \leq j \leq p-1} \nu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t_j).$$

Using the contraction condition repeatedly, we get

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_{j+1} - x_j, t_j) \leq \nu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, (1-\rho)\lambda\rho^{-k}\right).$$

Therefore,

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) \leq \nu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, (1-\rho)\lambda\rho^{-k}\right).$$

Again, since

$$(1-\rho)\lambda\rho^{-k} \to \infty \quad \text{as} \quad k \to \infty,$$

we obtain

$$\nu_F^3\left(z_1, \ldots, z_{n-1}, x_1 - x_0, (1-\rho)\lambda\rho^{-k}\right) \to 0.$$

Hence, there exists $k_2 \in \mathbb{N}$ such that

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) < \varepsilon$$

for all $p > k \geq k_2$.

Let

$$k_0 = \max\{k_1, k_2\}.$$

Then, for all $p, k \geq k_0$, we have

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) > 1 - \varepsilon$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_p - x_k, \lambda) < \varepsilon.$$

Therefore, by Definition 4.1, the sequence $\langle x_k \rangle$ is Cauchy in $\mathcal{A}$.

Since $\mathcal{A}$ is complete, there exists $u \in X$ such that

$$x_k \to u \quad \text{in } \mathcal{A}.$$

Now, we prove that u is a fixed point of τ. Since

$$x_{k+1} = \tau x_k,$$

we have

$$x_{k+1} - \tau u = \tau x_k - \tau u.$$

Using the contraction condition, for every $\lambda > 0$,

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_{k+1} - \tau u, \lambda) = \mu_F^3(z_1, \ldots, z_{n-1}, \tau x_k - \tau u, \lambda)$$
$$\geq \mu_F^3\left(z_1, \ldots, z_{n-1}, x_k - u, \frac{\lambda}{\rho}\right).$$

Since $x_k \to u$, the right-hand side tends to 1. Hence

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_{k+1} - \tau u, \lambda) \to 1.$$

Similarly,

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_{k+1} - \tau u, \lambda) \leq \nu_F^3\left(z_1, \ldots, z_{n-1}, x_k - u, \frac{\lambda}{\rho}\right),$$

and the right-hand side tends to 0. Therefore,

$$x_{k+1} \to \tau u.$$

On the other hand, since $x_k \to u$, the shifted sequence $\langle x_{k+1} \rangle$ also converges to u. By the uniqueness of the limit in Theorem 3.2, we obtain

$$\tau u = u.$$

Thus, u is a fixed point of τ.

It remains to prove uniqueness. Suppose that $u, v \in X$ are fixed points of τ. Then

$$\tau u = u \quad \text{and} \quad \tau v = v.$$

For every $\lambda > 0$, the contraction condition gives

$$\mu_F^3(z_1, \ldots, z_{n-1}, u - v, \lambda) = \mu_F^3(z_1, \ldots, z_{n-1}, \tau u - \tau v, \lambda)$$
$$\geq \mu_F^3\left(z_1, \ldots, z_{n-1}, u - v, \frac{\lambda}{\rho}\right).$$

Applying this repeatedly, we get

$$\mu_F^3(z_1, \ldots, z_{n-1}, u - v, \lambda) \geq \mu_F^3\left(z_1, \ldots, z_{n-1}, u - v, \frac{\lambda}{\rho^m}\right)$$

for every $m \in \mathbb{N}$. Since $\rho \in (0, 1)$,

$$\frac{\lambda}{\rho^m} \to \infty \quad \text{as} \quad m \to \infty.$$

Hence,

$$\mu_F^3\left(z_1, \ldots, z_{n-1}, u - v, \frac{\lambda}{\rho^m}\right) \to 1.$$

Therefore,

$$\mu_F^3(z_1, \ldots, z_{n-1}, u - v, \lambda) = 1.$$

Thus,

$$\mu_F(z_1, \ldots, z_{n-1}, u - v, \lambda) = 1.$$

By Definition 3.1, this means that $z_1, z_2, \ldots, z_{n-1}, u - v$ are linearly dependent. If $u \neq v$, then $u - v \neq 0$. We may choose $z_1, z_2, \ldots, z_{n-1} \in X$ such that

$$z_1, z_2, \ldots, z_{n-1}, u - v$$

are linearly independent. This contradicts the conclusion above. Hence,

$$u = v.$$

Therefore, τ has a unique fixed point in X. □

7. Application

In this section, we present an application of the fixed point theorem in a Fermatean intuitionistic fuzzy n-normed linear space. The main purpose is to show that the abstract fixed point result can be used to guarantee the existence and uniqueness of solutions of operator equations. In particular, we consider an affine operator equation of the form $x = c + \alpha x$, where $c \in X$ is fixed and α is a scalar satisfying $|\alpha| < 1$.

This type of equation can be rewritten as a fixed point problem $\tau(x) = x$, where $\tau : X \to X$ is defined by $\tau(x) = c + \alpha x$. Therefore, if τ satisfies the Fermatean intuitionistic fuzzy contraction condition, then the fixed point theorem ensures that the equation has a unique solution.

Theorem 7.1. *Let $X = \mathbb{R}^n$ and define the usual n-norm on X by*

$$\|v_1, v_2, \ldots, v_n\| = |\det(v_1, v_2, \ldots, v_n)|.$$

For $v_1, \ldots, v_n \in X$ and $\lambda > 0$, define

$$\mu_F(v_1, \ldots, v_n, \lambda) = \left(\frac{\lambda}{\lambda + \|v_1, \ldots, v_n\|}\right)^{\frac{1}{3}}$$

and

$$\nu_F(v_1,\dots,v_n,\lambda)=\left(\frac{\|v_1,\dots,v_n\|}{\lambda+\|v_1,\dots,v_n\|}\right)^{\frac{1}{3}}.$$

Let

$$a*b=ab \qquad \text{and} \qquad a\diamond b=a+b-ab.$$

Then, $\mathcal{A}=\left(X^n,\mu_F,\nu_F,*,\diamond\right)$ *is a complete Fermatean intuitionistic fuzzy n-normed linear space.*

Let $c\in X$ *be fixed and let* $\alpha\in\mathbb{R}$ *satisfy* $|\alpha|<1$. *Consider the operator equation* $x=c+\alpha x$. *Then, this equation has a unique solution in* X.

Proof. Define a mapping $\tau: X\to X$ by $\tau(x)=c+\alpha x, \qquad x\in X$. Then, the operator equation $x=c+\alpha x$ is equivalent to the fixed point equation $\tau(x)=x$.
We show that τ is a Fermatean intuitionistic fuzzy contraction mapping. Take

$$\varphi(t)=t^2, \qquad \psi(t)=t^2, \qquad t\in[0,1],$$

and choose

$$\xi=0, \qquad \eta=0.$$

Then (φ,ψ) is a Fermatean intuitionistic fuzzy cubic-control function.

Let $x,y\in X$, let $z_1,z_2,\dots,z_{n-1}\in X$, and let $\lambda>0$. Since

$$\tau x-\tau y=(c+\alpha x)-(c+\alpha y)=\alpha(x-y),$$

we have

$$\|z_1,z_2,\dots,z_{n-1},\tau x-\tau y\|=|\alpha|\,\|z_1,z_2,\dots,z_{n-1},x-y\|.$$

Put

$$D=\|z_1,z_2,\dots,z_{n-1},x-y\|.$$

Then,

$$\|z_1,z_2,\dots,z_{n-1},\tau x-\tau y\|=|\alpha|D.$$

Now,

$$\mu_F^3(z_1,z_2,\dots,z_{n-1},\tau x-\tau y,\lambda)=\frac{\lambda}{\lambda+|\alpha|D}.$$

Since $|\alpha|<1$, we have

$$|\alpha|D\le D.$$

Hence,

$$\lambda+|\alpha|D\le\lambda+D.$$

Therefore,

$$\frac{\lambda}{\lambda+|\alpha|D}\ge\frac{\lambda}{\lambda+D}.$$

That is,

$$\mu_F^3(z_1,z_2,\dots,z_{n-1},\tau x-\tau y,\lambda)\ge\mu_F^3(z_1,z_2,\dots,z_{n-1},x-y,\lambda).$$

Since $\xi=\eta=0$, this becomes

$$\mu_F^3(z_1,\dots,z_{n-1},\tau x-\tau y,\lambda)\ge\frac{\mu_F^3(z_1,\dots,z_{n-1},x-y,\lambda)}{1+\xi\left[1-\mu_F^3(z_1,\dots,z_{n-1},x-y,\lambda)\right]+\eta\varphi\left(\mu_F(z_1,\dots,z_{n-1},x-y,\lambda)\right)}.$$

Next,

$$\nu_F^3(z_1,z_2,\dots,z_{n-1},\tau x-\tau y,\lambda)=\frac{|\alpha|D}{\lambda+|\alpha|D}.$$

Since $|\alpha|<1$, we have

$$\frac{|\alpha|D}{\lambda+|\alpha|D}\le\frac{D}{\lambda+D}.$$

Thus,

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \leq \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda).$$

Again, since $\xi = \eta = 0$, this is equivalent to

$$\nu_F^3(z_1, z_2, \ldots, z_{n-1}, \tau x - \tau y, \lambda) \leq \frac{\nu_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)}{1 + \xi \nu_F^3(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda) + \eta\psi\left(\nu_F(z_1, z_2, \ldots, z_{n-1}, x - y, \lambda)\right)}.$$

Therefore, τ is a Fermatean intuitionistic fuzzy (φ, ψ)-contraction mapping on $\mathcal{A}$.

Since $\mathcal{A}$ is complete, by the fixed point theorem, τ has a unique fixed point $x^* \in X$. Hence,

$$\tau(x^*) = x^*.$$

Therefore,

$$c + \alpha x^* = x^*.$$

Thus,

$$(1 - \alpha)x^* = c.$$

Since $|\alpha| < 1$, we have $\alpha \neq 1$, and hence

$$x^* = \frac{c}{1 - \alpha}.$$

Therefore, the equation $x = c + \alpha x$ has a unique solution in X, namely $x^* = \frac{c}{1-\alpha}$. □

This application shows that the fixed point theorem in a Fermatean intuitionistic fuzzy n-normed linear space can be used to guarantee the existence and uniqueness of solutions of operator equations of the form

$$x = \tau(x).$$

In particular, the affine equation

$$x = c + \alpha x$$

has a unique solution whenever $|\alpha| < 1$.

Example 7.1. *Let $X = \mathbb{R}^2$ and define the usual 2-norm on X by $\|u, v\| = \left|\det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\right|$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are elements of $\mathbb{R}^2$.*

For $u, v \in X$ and $\lambda > 0$, define

$$\mu_F(u, v, \lambda) = \left(\frac{\lambda}{\lambda + \|u, v\|}\right)^{\frac{1}{3}}$$

and

$$\nu_F(u, v, \lambda) = \left(\frac{\|u, v\|}{\lambda + \|u, v\|}\right)^{\frac{1}{3}}.$$

Let

$$a * b = ab, \qquad a \diamond b = a + b - ab,$$

for all $a, b \in [0, 1]$. Then

$$\mathcal{A} = \left(X^2, \mu_F, \nu_F, *, \diamond\right)$$

is a Fermatean intuitionistic fuzzy 2-normed linear space.

Define the mapping

$$\tau : X \to X$$

by

$$\tau(x) = c + \alpha x,$$

where

$$c = (2, 4) \qquad \text{and} \qquad \alpha = \frac{1}{2}.$$

Thus,

$$\tau(x) = (2,4) + \frac{1}{2}x.$$

Since

$$|\alpha| = \frac{1}{2} < 1,$$

the mapping τ is a contraction mapping.

The fixed point x^* of τ satisfies

$$\tau(x^*) = x^*.$$

Hence,

$$(2,4) + \frac{1}{2}x^* = x^*.$$

Therefore,

$$\frac{1}{2}x^* = (2,4),$$

and so

$$x^* = (4,8).$$

Thus, the unique fixed point of τ is x^*=(4,8). Indeed,

$$\tau(4,8) = (2,4) + \frac{1}{2}(4,8) = (2,4) + (2,4) = (4,8).$$

Therefore,

$$\tau(x^*) = x^*.$$

Now, we verify the contraction behavior numerically. Take

$$x = (0,0), \qquad y = (2,1), \qquad z = (1,3), \qquad \lambda = 2.$$

Then,

$$x - y = (-2,-1).$$

Hence,

$$\|z, x-y\| = \left|\det \begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix}\right| = |-1+6| = 5.$$

Therefore,

$$\mu_F^3(z, x-y, 2) = \frac{2}{2+5} = \frac{2}{7},$$

and

$$\nu_F^3(z, x-y, 2) = \frac{5}{2+5} = \frac{5}{7}.$$

Next,

$$\tau(x) = (2,4) + \frac{1}{2}(0,0) = (2,4),$$

and

$$\tau(y) = (2,4) + \frac{1}{2}(2,1) = \left(3, \frac{9}{2}\right).$$

Thus,

$$\tau(x) - \tau(y) = (2,4) - \left(3, \frac{9}{2}\right) = \left(-1, -\frac{1}{2}\right).$$

Hence,

$$\|z, \tau(x) - \tau(y)\| = \left|\det \begin{pmatrix} 1 & -1 \\ 3 & -\frac{1}{2} \end{pmatrix}\right| = \left|-\frac{1}{2} + 3\right| = \frac{5}{2}.$$

Therefore,

$$\mu_F^3(z, \tau(x) - \tau(y), 2) = \frac{2}{2+\frac{5}{2}} = \frac{4}{9},$$

and

$$\nu_F^3(z, \tau(x) - \tau(y), 2) = \frac{\frac{5}{2}}{2 + \frac{5}{2}} = \frac{5}{9}.$$

Now, we observe that

$$\frac{4}{9} > \frac{2}{7},$$

so

$$\mu_F^3(z, \tau(x) - \tau(y), 2) > \mu_F^3(z, x - y, 2).$$

Also,

$$\frac{5}{9} < \frac{5}{7},$$

so

$$\nu_F^3(z, \tau(x) - \tau(y), 2) < \nu_F^3(z, x - y, 2).$$

Thus, under the mapping τ, the membership degree increases and the non-membership degree decreases. This confirms the contraction behavior in the Fermatean intuitionistic fuzzy 2-normed linear space.

Finally, starting from $x_0 = (0, 0)$, the fixed point iteration $x_{k+1} = \tau(x_k)$ gives

$$x_1 = \tau(x_0) = (2, 4),$$
$$x_2 = \tau(x_1) = (3, 6),$$
$$x_3 = \tau(x_2) = \left(\frac{7}{2}, 7\right),$$
$$x_4 = \tau(x_3) = \left(\frac{15}{4}, \frac{15}{2}\right),$$

and, in general, $x_k \to (4, 8)$. Hence, numerically, τ has the unique fixed point (4,8).

8. Conclusion

In this paper, we introduced the concept of Fermatean intuitionistic fuzzy n-normed linear spaces as a new generalization of intuitionistic fuzzy n-normed linear spaces. The proposed framework combines the geometric structure of n-normed linear spaces with the Fermatean cubic admissibility condition

$$\mu_F^3 + \nu_F^3 \leq 1.$$

This condition allows a wider range of membership and non-membership degrees than the classical intuitionistic fuzzy condition

$$\mu + \nu \leq 1,$$

and therefore provides a more flexible structure for handling uncertainty in n-normed linear spaces.

Several basic properties of the proposed space were developed. We introduced the Fermatean hesitation degree, open balls, convergence, Cauchy sequences, and completeness in the setting of FIF-n-NLS. Equivalent characterizations of convergence and Cauchy sequences were obtained in terms of the limiting behavior of the Fermatean membership and non-membership functions. In particular, convergence was characterized by

$$\mu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) \to 1$$

and

$$\nu_F^3(z_1, \ldots, z_{n-1}, x_k - x, \lambda) \to 0,$$

while the Cauchy property was characterized by the corresponding limits for $x_k - x_p$. We also proved that the limit of a convergent sequence is unique and that every convergent sequence is Cauchy.

Furthermore, we introduced Fermatean intuitionistic fuzzy cubic-control functions and Fermatean intuitionistic fuzzy contraction mappings. Using these notions, a Banach-type fixed point theorem was established in complete Fermatean intuitionistic fuzzy n-normed linear spaces. This result extends the classical fixed point principle to the Fermatean intuitionistic fuzzy n-normed setting and is consistent with earlier developments in fixed point theory on fuzzy and intuitionistic fuzzy normed structures [48, 32, 35]. Finally, an application was given to show that the proposed fixed point theorem can guarantee the existence and uniqueness of solutions of a linear operator equation.

The results obtained in this paper show that Fermatean intuitionistic fuzzy n-normed linear spaces form a meaningful and useful extension of both intuitionistic fuzzy n-normed linear spaces and Fermatean fuzzy structures [42, 21]. The examples presented throughout the paper demonstrate that the proposed definitions are mathematically consistent and applicable to concrete spaces such as $\mathbb{R}^n$.

9. Suggestions and Further Work

The present work provides only the foundational theory of Fermatean intuitionistic fuzzy n-normed linear spaces. Several directions may be considered for future research.

First, one may study different types of convergence in FIF-n-NLS, such as statistical convergence, ideal convergence, lacunary statistical convergence, and summability methods. Similar ideas have already been investigated in intuitionistic fuzzy normed and intuitionistic fuzzy n-normed spaces [36, 44, 46], and it would be natural to extend them to the Fermatean intuitionistic fuzzy setting.

Second, the fixed point results developed in this paper can be generalized to other types of contractive mappings. For example, future studies may consider Kannan-type, Chatterjea-type, Reich-type, Hardy–Rogers-type, and Ciric-type contractions in FIF-n-NLS. Such extensions may lead to broader fixed point theorems and further applications to nonlinear operator equations.

Third, the role of cubic-control functions can be investigated more deeply. Different choices of the control functions φ and ψ may produce new classes of contractions. It would also be interesting to study whether these control functions can be replaced by altering-distance functions, simulation functions, or comparison functions in the sense of generalized fixed point theory.

Fourth, since Fermatean fuzzy sets provide a wider admissible region than intuitionistic fuzzy and Pythagorean fuzzy sets [17, 18, 21], the proposed FIF-n-NLS may be applied to decision-making problems, uncertainty modelling, optimization, and approximation theory. In particular, one may develop Fermatean intuitionistic fuzzy versions of operator approximation theorems, stability results, and Korovkin-type approximation theorems.

Finally, further research may also focus on applications of FIF-n-NLS to differential equations, integral equations, control problems, and dynamic systems under uncertainty. Since fixed point theory is widely used to prove existence and uniqueness of solutions, the Banach-type theorem established in this paper may serve as a useful tool for future analytical and applied investigations.

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